

Another Approach to the Brewer-Dobson-Circulation: the direct inversion of the Continuity Equation:

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$$\frac{\partial \frac{\partial v m r_{i+1}(\phi, z)}{\partial v m r_i(\phi, z)}}{\partial v(\phi, z)} = - \left(\frac{\Delta t_p}{\Delta \phi} \right)^2 \cdot \frac{1}{r^2} \cdot \left(v(\phi, z) + \frac{v(\phi - \Delta \phi, z)}{2} \right) \quad (46)$$

$$\frac{\partial \frac{\partial v m r_{i+1}(\phi + \Delta \phi, z)}{\partial v m r_i(\phi + \Delta \phi, z)}}{\partial v(\phi, z)} = - \left(\frac{\Delta t_p}{\Delta \phi} \right)^2 \cdot \frac{v(\phi + \Delta \phi, z)}{2r^2} \quad (47)$$

$$\frac{\partial \frac{\partial v m r_{i+1}(\phi, z)}{\partial v m r_i(\phi - \Delta \phi, z)}}{\partial v(\phi, z)} = \frac{1}{2r} \cdot \frac{\Delta t_p}{\Delta \phi} \cdot \left(1 + \frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi - \Delta \phi, z)}{r} \right) \quad (48)$$

$$\frac{\partial \frac{\partial v m r_{i+1}(\phi + \Delta \phi, z)}{\partial v m r_i(\phi, z)}}{\partial v(\phi, z)} = \frac{1}{2r} \left(\frac{\Delta t_p}{\Delta \phi} \right)^2 \frac{v(\phi + \Delta \phi, z)}{r} \quad (49)$$

$$\frac{\partial \frac{\partial v m r_{i+1}(\phi, z)}{\partial v m r_i(\phi + \Delta \phi, z)}}{\partial v(\phi, z)} = - \frac{\Delta t_p}{\Delta \phi} \cdot \frac{1}{r} \left(\frac{1}{2} - \frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi, z)}{r} \right) \quad (50)$$

Entries not mentioned here are zero. Entries relevant to w are:

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi, z)}{\partial \rho_i(\phi, z)}}{\partial w(\phi, z)} = - \frac{\Delta t_p}{\Delta \phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{v(\phi, z)}{r} \quad (51)$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi, z)}{\partial \rho_i(\phi, z + \Delta z)}}{\partial w(\phi, z)} = \frac{1}{2} \cdot \left(\frac{\Delta t_p}{\Delta z} \right)^2 \cdot \frac{(r + \Delta z)^2}{r^2} \cdot w(\phi, z + \Delta z) \quad (56)$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi, z - \Delta z)}{\partial \rho_i(\phi, z)}}{\partial w(\phi, z)} = \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{r^2}{(r - \Delta z)^2} \cdot \left(-1 + \frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi, z - \Delta z)}{r - \Delta z} + \frac{\Delta t_p}{\Delta z} \cdot w(\phi, z - \Delta z) \right) \quad (57)$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi + \Delta \phi, z - \Delta z)}{\partial \rho_i(\phi, z)}}{\partial w(\phi, z)} = - \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta \phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{\cos(\phi)}{r^2} \cdot \frac{v(\phi, z - \Delta z)}{\cos(\phi + \Delta \phi) \cdot r - \Delta z} \quad (58)$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi, z + \Delta z)}{\partial \rho_i(\phi + \Delta \phi, z)}}{\partial w(\phi, z)} = - \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta \phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{\cos(\phi + \Delta \phi)}{r^2} \cdot \frac{v(\phi + \Delta \phi, z)}{\cos(\phi) \cdot r} \quad (59)$$

$$\frac{\partial \frac{\partial v m r_{i+1}(\phi, z)}{\partial v m r_i(\phi, z)}}{\partial w(\phi, z)} = - \left(\frac{\Delta t_p}{\Delta z} \right)^2 \cdot \left(w(\phi, z) + \frac{w(\phi, z - \Delta z)}{2} \right) \quad (60)$$

The working title of this project is

ANCISTRUS SPEC.

What is an ANCISTRUS SPEC?

What is an ANCISTRUS SPEC.?



And this is our model of it:

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==> Starting SunOS Version 0.4.0 on host imksuns18 <==
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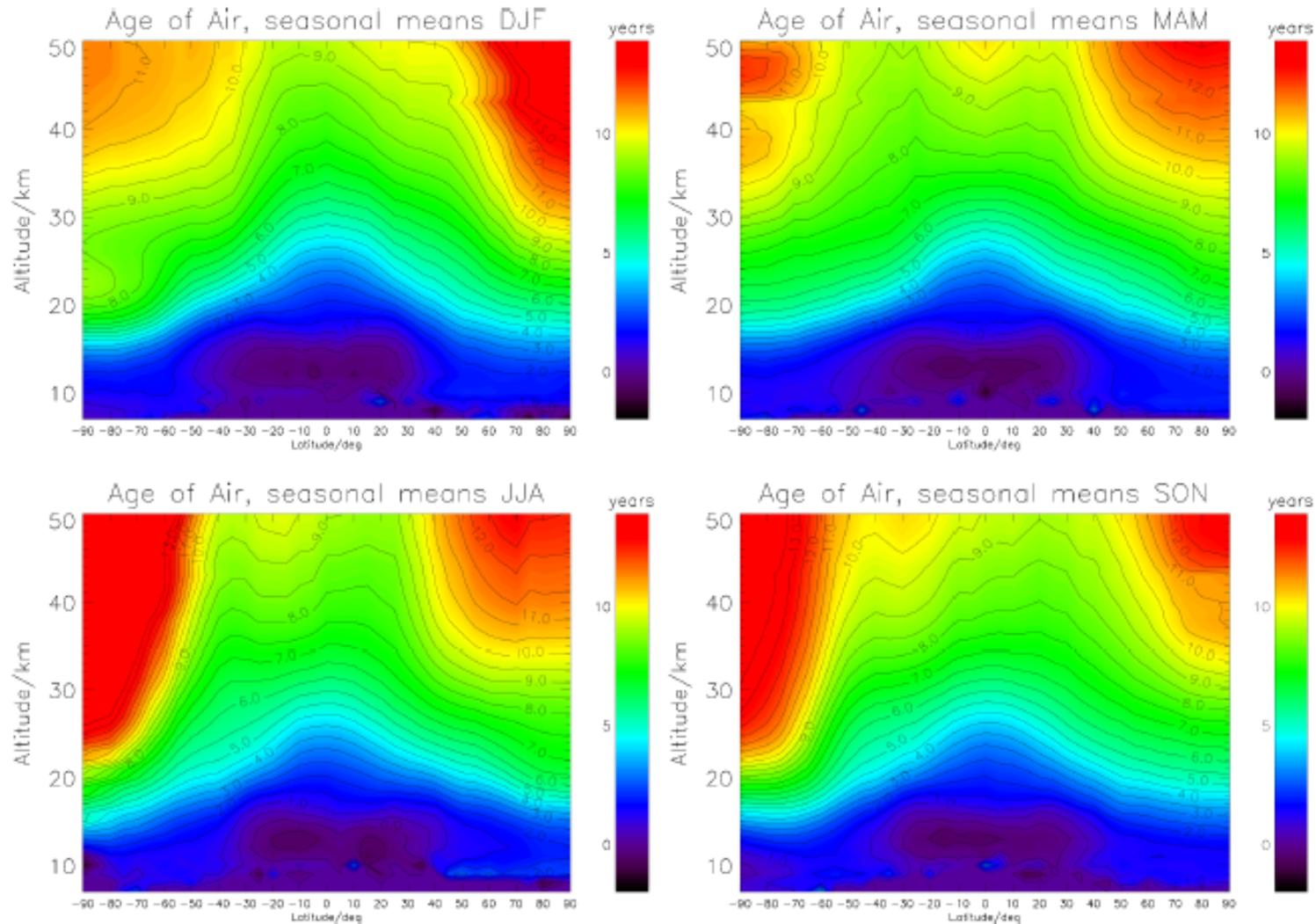
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ANalysis of CIRCulation in the STRatosphere USING
SPECTroscopic measurements of tracers
0.4.0 / 05-Nov-2015
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using Processor library version 1.6.12

Usually, atmospheric circulation is analyzed via the **mean age of stratospheric air**.



F. Haenel
et al.
ACP,
2015

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- Mean age of stratospheric air.
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All these methods have their own problems!

- Mean age of stratospheric air.
 - The **age spectrum** must be known.
 - Tropospheric **reference data** are needed for the past.
 - Subsidence of mesospheric air into the stratospheric polar vortex and subsequent in-mixing in mid-latitudinal air can cause **an old bias**.
- Correlation analysis
 - Structured patterns are needed;
 - Reduced spatial resolution
- Flux Analysis
 - Only local (non-global) interactive method

All these methods have their own problems!

- The age of air can be understood as an “**integral** over the stratospheric residence time of the air parcel”. Roughly speaking, the intensity of the circulation is the quotient of the displacement of the air parcel during its stratospheric life and its stratospheric residence time.
- If we find an approach which is more **differential**, we can avoid the problems mentioned before: The strength of the circulation then is the quotient of a short-time displacement and the related time-step.
- The **continuity equation** helps to achieve this!

2-D continuity equation:

- Local change of air density:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{r} \frac{\partial \rho v}{\partial \phi} + \frac{\rho v}{r} \tan(\phi) - \frac{\partial \rho w}{\partial z} - \frac{2\rho w}{r}$$

- Local change of mixing ratio:

$$\begin{aligned} \frac{\partial v m r_g}{\partial t} = & \frac{S_g}{\rho} - \frac{v}{r} \frac{\partial v m r_g}{\partial \phi} - w \frac{\partial v m r_g}{\partial z} \\ & + \frac{1}{(r)^2 \cos \phi} \frac{\partial}{\partial \phi} \left[K_\phi \cos \phi \frac{\partial v m r_g}{\partial \phi} \right] \\ & + \frac{1}{r^2} \frac{\partial}{\partial z} \left[r^2 K_z \frac{\partial v m r_g}{\partial z} \right] \end{aligned}$$

V and W are the velocities plus contribution by the anti-symmetric part of the eddy flux tensor.

K_ϕ and K_z are the mixing coefficients plus the contribution of the symmetric part of the eddy flux tensor (c.f. Ko et al., JGR 1985)

2-D continuity equation

- Usual transport calculations:
(vmr, density) = continuity equation (v, w, K_ϕ, K_z ; $\text{vmr}_0, \text{density}_0$)
- What we do:
(v, w, K_ϕ, K_z) = continuity equation $^{-1}$ (vmr, density, $\text{vmr}_0, \text{density}_0$)

For us, tracer analysis is the inverse solution of the continuity equation.

The major steps

1. Predict the later state for a given initial state by the 'forward solution' of the continuity equation for assumed kinematic variables (transport calculation).
2. Estimate the uncertainties of the predicted later state (error estimation).
3. Calculate the sensitivity of the later state with respect to the kinematic variables (Jacobian matrix).
4. Compare the predicted later state with the measured later state and calculate the difference (residual).
5. Invert the continuity equation by minimization of the residual.
6. The resulting kinematic variables represent the most likely velocities and mixing coefficients

Step 1: Transport modeling:

- We integrate the tendencies:
new state = old state + $\delta\text{state}/\delta\text{kinematic} * \text{kinematic}$
- Simple Euler integration is too diffusive.
- State of the art integration (e.g. Prather) does not allow a closed form solution for the Jacobians.
- Compromise: MacCormack-integration: a predictor-corrector method.

$$c_{i+1}^*(x,y) = c_i(x,y) - \frac{\Delta t_p}{\Delta x} (e_i(x + \Delta x, y) - e_i(x, y)) - \frac{\Delta t_p}{\Delta y} (f_i(x, y + \Delta y) - f_i(x, y)).$$

$$c_{i+1}(x,y) = \frac{1}{2} [c_i(x,y) + c_{i+1}^*(x,y) - \frac{\Delta t_p}{\Delta x} (e(c_{i+1}^*, x, y) - e(c_{i+1}^*, x - \Delta x, y)) - \frac{\Delta t_p}{\Delta y} (f(c_{i+1}^*, x, y) - f(c_{i+1}^*, x, y - \Delta y))]$$

Step 2: Error Propagation

It is convenient to write the transport problem in matrix notation:

$$\text{Later state} = D * \text{initial state} = D_n * \dots * D_2 * D_1 * \text{initial state};$$

Each $D_{i,i=1,n}$ refers to one ‘micro-timestep’;

In this case, the prediction error S_1 is calculated from the uncertainty of the initial state S_0 via generalized Gaussian error propagation as

$$S_1 = \Pi D_n S_0 (\Pi D_n)^T ;$$

The “only” problem is to get the D_n matrices 😊

locations. The sensitivities of the densities of the first predictive step with respect to the initial densities at the same latitude and altitude are

$$\frac{\partial \rho_{i+1}(\phi, z)}{\partial \rho_i(\phi, z)} = \frac{1}{2} \left[2 - \frac{\Delta t_p}{\Delta \phi} \left[\frac{v(\phi, z)}{r} \left(\frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi, z)}{r} + \frac{\Delta t_p}{\Delta z} w(\phi, z) \right) + \frac{\Delta t_p}{\Delta \phi} \frac{v(\phi - \Delta \phi, z) v(\phi, z)}{r^2} \right] - \frac{\Delta t_p}{\Delta z} \left[w(\phi, z) \left(\frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi, z)}{r} + \frac{\Delta t_p}{\Delta z} w(\phi, z) \right) + \frac{\Delta t_p}{\Delta z} w(\phi, z - \Delta z) w(\phi, z) \right] \right] \quad (15)$$

We further differentiate predicted air densities with respect to air densities at the adjacent southern latitude but the same altitude.

$$\frac{\partial \rho_{i+1}(\phi, z)}{\partial \rho_i(\phi - \Delta \phi, z)} = \frac{1}{2} \left[\frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi - \Delta \phi, z)}{r} \cdot \frac{\cos(\phi - \Delta \phi)}{\cos(\phi)} \left(1 + \frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi - \Delta \phi, z)}{r} + \frac{\Delta t_p}{\Delta z} w(\phi - \Delta \phi, z) \right) \right] \quad (16)$$

The derivative of the predicted air densities with respect to air densities at the adjacent northern latitude but the same altitude is

$$\frac{\partial \rho_{i+1}(\phi, z)}{\partial \rho_i(\phi - \Delta \phi, r + \Delta z)} = \frac{1}{2} \left[\frac{v(\phi - \Delta \phi, z)}{r} \cdot \frac{\Delta t_p}{\Delta \phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{(r + \Delta z)^2}{r^2} \cdot \frac{\cos(\phi - \Delta \phi)}{\cos(\phi)} w(\phi - \Delta \phi, r + \Delta z) \right] \quad (20)$$

and vice versa

$$\frac{\partial \rho_{i+1}(\phi, z)}{\partial \rho_i(\phi + \Delta \phi, r - \Delta z)} = -\frac{1}{2} \left[w(\phi, r - \Delta z) \frac{\Delta t_p}{\Delta z} \cdot \frac{\Delta t_p}{\Delta \phi} \cdot \frac{(r - \Delta z)^2}{r^2} \cdot \frac{\cos(\phi + \Delta \phi)}{\cos(\phi)} \cdot \frac{v(\phi + \Delta \phi, z - \Delta z)}{r - \Delta z} \right]$$

where i is the index of the time increment, $z \pm \Delta z$ refer to the adjacent model grid and altitude, respectively.

For mixing ratios, the respective derivative

$$\frac{\partial \text{vmr}_{i+1}(\phi, z)}{\partial \text{vmr}_i(\phi, z)} = 1 - \left(\frac{\Delta t_p}{\Delta \phi} \right)^2 \cdot \frac{v(\phi, z)}{r^2} \cdot \frac{1}{2} [v(\phi, z) + v(\phi - \Delta \phi, z)] - \left(\frac{\Delta t_p}{\Delta z} \right)^2 \cdot \frac{1}{2} [w(\phi, z) + w(\phi - \Delta \phi, z)]$$

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$$\frac{\partial \text{vmr}_{i+1}(\phi, z)}{\partial \text{vmr}_i(\phi + \Delta \phi, z)} = -\frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi, z)}{2r} \left(1 - \frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi, z)}{r} \right) + \frac{\Delta t_p}{2r^2(\Delta \phi)^2 \cos(\phi)} \cdot \left(K_\phi(\phi, z) + K_\phi(\phi + \Delta \phi, z) \right) \cos(\phi + \frac{\Delta \phi}{2}); \quad (23)$$

$$\frac{\partial \text{vmr}_{i+1}(\phi, z)}{\partial \text{vmr}_i(\phi - \Delta \phi, z)} = \frac{v(\phi, z)}{2r} \cdot \frac{\Delta t_p}{\Delta \phi} \left(1 + \frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi - \Delta \phi, z)}{r} \right) + \frac{\Delta t_p}{2r^2(\Delta \phi)^2 \cos(\phi)} \cdot \left(K_\phi(\phi, z) + K_\phi(\phi - \Delta \phi, z) \right) \cos(\phi - \frac{\Delta \phi}{2}); \quad (24)$$

$$\frac{\partial \text{vmr}_{i+1}(\phi, z)}{\partial \text{vmr}_i(\phi, z + \Delta z)} = -\frac{1}{2} \cdot w(\phi, z) \cdot \frac{\Delta t_p}{\Delta z} \left(1 - \frac{(\Delta t_p)}{(\Delta z)} w(\phi, z) \right) + \frac{\Delta t_p}{2r^2(\Delta z)^2} \left(r + \frac{\Delta z}{2} \right)^2 \left(K_z(\phi, z) + K_z(\phi, z + \Delta z) \right); \quad (25)$$

$$\frac{\partial \text{vmr}_{i+1}(\phi, z)}{\partial \text{vmr}_i(\phi, z - \Delta z)} = \frac{\Delta t_p}{\Delta z} \cdot \frac{1}{2} \cdot w(\phi, z) \left(1 + w(\phi, z - \Delta z) \frac{\Delta t_p}{\Delta z} \right) + \frac{\Delta t_p}{2r^2(\Delta z)^2} \left(r - \frac{\Delta z}{2} \right)^2 \left(K_z(\phi, z) + K_z(\phi, z - \Delta z) \right); \quad (26)$$

These derivatives are simplifications in a sense that they do not consider the full chemical Jacobian but assume instead that the source strength depends on no other concentration than the actual concentration of the same species. For the typical long-lived so-called tropospheric source gases considered here, like SF₆ or CFCs, this assumption is appropriate.

With these expressions, the prediction of air density and volume mixing ratio can be rewritten in matrix notation for a

T. von Clarmann et al.: Circulation and Mixing from Tracer Measurements

where $\mathbf{D}_{\rho,i}$ is the $L_0 \times L_0$ Jacobian matrix of air density for time increment i , i.e. the sensitivities of the prediction with respect to the initial state, $\frac{\partial c_{i+1,m}}{\partial c_{i,n}}$, here m and n run over the model gridpoints
 \mathbf{I}_K is $K_0 \times K_0$ identity;
 $\mathbf{0}$ are zero submatrices of the required dimensions;
 \mathbf{W}_i is a $K_0 \times 2K_0$ -dimensional interpolation matrix;
 $\mathbf{D}_{\rho,j}$ is an $J_0 \times L_0$ Jacobian containing the partial derivatives $\partial \rho_{i+1,j} / \partial \rho_{i,l}$;
 $\rho_{i,k=1,K_0}$ is the K_0 -dimensional vector of air densities in the border region after the final timestep, i.e. for the time of the next measurement;
 $\rho_{i,k=1,K_0}$ is the K_0 -dimensional vector of air densities in the border region at the current timestep as resulting from interpolation in time;
 $\rho_{i,j=K_0+1,L_0}$ is the K_0 -dimensional vector of air densities in the nominal region at the current timestep as resulting from integration according to the MacCormack scheme as described above.

Since the source term depends on air density, the integration in matrix notation for vmr requires simultaneous treatment of vmr and air density and we get, using notation according to the previous section, the following matrix notation with air density:

$$\begin{pmatrix} \rho_{i+1} \\ \text{vmr}_{i+1} \end{pmatrix} = \begin{pmatrix} \rho_{i=1} \\ \vdots \\ \rho_{L_0} \\ \text{vmr}_{g;l=1} \\ \vdots \\ \text{vmr}_{g;\sum L_g} \end{pmatrix} = \mathbf{D}_i \begin{pmatrix} \rho_i \\ \text{vmr}_{g;i} \end{pmatrix} = \begin{pmatrix} \mathbf{D}_{\rho,i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_K & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{g,j} \end{pmatrix} \begin{pmatrix} \rho_{i,l=1,L_0} \\ \text{vmr}_{g;i,k=1,K_g} \\ \text{vmr}_{g;i,k=1,K_g} \\ \text{vmr}_{g;i,j=K_g+1,L_g} \end{pmatrix}$$

where \mathbf{D}_i is the total Jacobian with respect to air density and all involved gas mixing ratios. Note that

3.Step: Jacobian

- The Jacobian is $(\delta \text{ new state})/(\delta \text{ kinematic variables})$

Recall: state: mixing ratios, air densities

kinematic variables: pseudo-velocities, mixing coefficients.

This requires again a little bit of maths 😊

efficiently computed using the following recursive scheme, where $\tilde{f}_{l,i}$ is the respective column of the Jacobian after micro timestep i :

$$\tilde{f}_{n,i} = D_i \tilde{f}_{n,i-1} + \frac{\partial D_i}{\partial q_n} \left(\prod_{k=i-1}^1 D_k \right) \tilde{x}_0 \quad (36)$$

With the argument of D specifying the column of the D -matrix such that $D_{c,i}(\phi, z)$ relates $\rho_{i+1}(\phi, z)$ to $\rho_i(\phi, z)$, $D_{\rho,i}(\phi \pm \Delta\phi, z)$ relates $\rho_{i+1}(\phi, z)$ to $\rho_i(\phi \pm \Delta\phi, z)$, and $D_{\rho,i}(\phi, z \pm \Delta z)$ relates $\rho_{i+1}(\phi, z)$ to $\rho_i(\phi, z \pm \Delta z)$, and for v accordingly, the entries of D_i relevant to v are:

$$\frac{\partial \theta_{\rho_{i+1}(\phi, z)}}{\partial v(\phi, z)} = -\frac{\Delta t_p}{2\Delta\phi} \cdot \left(\frac{\Delta t_p}{\Delta\phi} \cdot \frac{2v(\phi, z) + v(\phi - \Delta\phi, z)}{r^2} + 2\frac{\Delta t_p}{\Delta z} \cdot \frac{w(\phi, z)}{r} \right) \quad (37)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi + \Delta\phi, z)}}{\partial v(\phi, z)} = -\frac{1}{2} \cdot \left(\frac{\Delta t_p}{\Delta\phi} \right)^2 \cdot \frac{v(\phi + \Delta\phi, z)}{r^2} \quad (38)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi + \Delta\phi, z)}}{\partial v(\phi, z)} = \frac{1}{2r} \cdot \frac{\Delta t_p}{\Delta\phi} \cdot \frac{\cos(\phi)}{\cos(\phi + \Delta\phi)} \cdot \left(1 + 2\frac{\Delta t_p}{\Delta\phi} \cdot \frac{v(\phi, z)}{r} + \frac{\Delta t_p}{\Delta z} w(\phi, z) \right) \quad (39)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi, z)}}{\partial v(\phi, z)} = \frac{1}{2} \cdot \left(\frac{\Delta t_p}{\Delta\phi} \right)^2 \cdot \frac{v(\phi + \Delta\phi, z)}{r^2} \cdot \frac{\cos(\phi + \Delta\phi)}{\cos(\phi)} \quad (40)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi - \Delta\phi, z)}}{\partial v(\phi, z)} = \frac{1}{2r} \cdot \frac{\Delta t_p}{\Delta\phi} \cdot \frac{\cos(\phi)}{\cos(\phi - \Delta\phi)} \cdot \left(-1 + \frac{\Delta t_p}{\Delta\phi} \cdot \frac{v(\phi - \Delta\phi, z)}{r} + \frac{\Delta t_p}{\Delta z} w(\phi - \Delta\phi, z) \right) \quad (41)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi, z + \Delta z)}}{\partial v(\phi, z)} = \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{\Delta t_p}{\Delta\phi} \cdot \frac{r^2}{(r + \Delta z)^2} \cdot \frac{w(\phi, z)}{r} \quad (42)$$

4)

$$\frac{\partial \theta_{\rho_{i+1}(\phi, z + \Delta z)}}{\partial v(\phi, z)} = \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{\Delta t_p}{\Delta\phi} \cdot \frac{(r + \Delta z)^2}{r^3} \cdot w(\phi, z + \Delta z) \quad (43)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi + \Delta\phi, z)}}{\partial v(\phi, z)} = -\frac{1}{2} \cdot \frac{\Delta t_p}{\Delta\phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{(r + \Delta z)^2}{r^3} \cdot \frac{\cos(\phi)}{\cos(\phi + \Delta\phi)} w(\phi, z + \Delta z) \quad (44)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi - \Delta\phi, z + \Delta z)}}{\partial v(\phi, z)} = -\frac{1}{2} \cdot \frac{\Delta t_p}{\Delta\phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{r}{(r + \Delta z)^2} \cdot \frac{\cos(\phi)}{\cos(\phi - \Delta\phi)} \cdot w(\phi - \Delta\phi, z) \quad (45)$$

$$\frac{\partial \theta_{v_{i+1}(\phi, z)}}{\partial v(\phi, z)} = -\left(\frac{\Delta t_p}{\Delta\phi} \right)^2 \cdot \frac{1}{r^2} \cdot \left(v(\phi, z) + \frac{v(\phi - \Delta\phi, z)}{2} \right) \quad (46)$$

$$\frac{\partial \theta_{v_{i+1}(\phi + \Delta\phi, z)}}{\partial v(\phi, z)} = -\left(\frac{\Delta t_p}{\Delta\phi} \right)^2 \cdot \frac{v(\phi + \Delta\phi, z)}{2r^2} \quad (47)$$

$$\frac{\partial \theta_{v_{i+1}(\phi - \Delta\phi, z)}}{\partial v(\phi, z)} = \frac{1}{2r} \cdot \frac{\Delta t_p}{\Delta\phi} \cdot \left(1 + \frac{\Delta t_p}{\Delta\phi} \cdot \frac{v(\phi - \Delta\phi, z)}{r} \right) \quad (48)$$

$$\frac{\partial \theta_{v_{i+1}(\phi + \Delta\phi, z)}}{\partial v(\phi, z)} = \frac{1}{2r} \cdot \left(\frac{\Delta t_p}{\Delta\phi} \right)^2 \cdot \frac{v(\phi + \Delta\phi, z)}{r} \quad (49)$$

$$\frac{\partial \theta_{v_{i+1}(\phi + \Delta\phi, z)}}{\partial v(\phi, z)} = -\frac{\Delta t_p}{\Delta\phi} \cdot \frac{1}{r} \cdot \left(1 - \frac{\Delta t_p}{\Delta\phi} \cdot \frac{v(\phi, z)}{r} \right) \quad (50)$$

Entries not mentioned here are zero. Entries relevant to w are:

$$\frac{\partial \theta_{\rho_{i+1}(\phi, z)}}{\partial w(\phi, z)} = -\frac{\Delta t_p}{\Delta\phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{v(\phi, z)}{r} - \left(\frac{\Delta t_p}{\Delta z} \right)^2 w(\phi, z) - \frac{1}{2} \left(\frac{\Delta t_p}{\Delta z} \right)^2 w(\phi, z - \Delta z) \quad (51)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi, z + \Delta z)}}{\partial w(\phi, z)} = -\frac{1}{2} \left(\frac{\Delta t_p}{\Delta z} \right)^2 w(\phi, z + \Delta z) \quad (52)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi, z)}}{\partial w(\phi, z)} = \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta\phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{v(\phi + \Delta\phi, z)}{r} \cdot \frac{\cos(\phi + \Delta\phi)}{\cos(\phi)} \quad (53)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi + \Delta\phi, z)}}{\partial w(\phi, z)} = \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta\phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{v(\phi, z)}{r} \cdot \frac{\cos(\phi)}{\cos(\phi + \Delta\phi)} \quad (54)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi, z + \Delta z)}}{\partial w(\phi, z)} = \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{r^2}{(r + \Delta z)^2} \cdot \left(1 + \frac{\Delta t_p}{\Delta\phi} \cdot \frac{v(\phi, z)}{r} + 2\frac{\Delta t_p}{\Delta z} \cdot w(\phi, z) \right) \quad (55)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi, z)}}{\partial w(\phi, z)} = \frac{1}{2} \cdot \left(\frac{\Delta t_p}{\Delta z} \right)^2 \cdot \frac{(r + \Delta z)^2}{r^2} \cdot w(\phi, z + \Delta z) \quad (56)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi, z - \Delta z)}}{\partial w(\phi, z)} = \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{r^2}{(r - \Delta z)^2} \cdot \left(-1 + \frac{\Delta t_p}{\Delta\phi} \cdot \frac{v(\phi, z - \Delta z)}{r - \Delta z} + \frac{\Delta t_p}{\Delta z} \cdot w(\phi, z - \Delta z) \right) \quad (57)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi + \Delta\phi, z - \Delta z)}}{\partial w(\phi, z)} = -\frac{1}{2} \cdot \frac{\Delta t_p}{\Delta\phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{r^2}{(r - \Delta z)^2} \cdot \frac{\cos(\phi)}{\cos(\phi + \Delta\phi)} \cdot \frac{v(\phi, z - \Delta z)}{r - \Delta z} \quad (58)$$

$$\frac{\partial \theta_{\rho_{i+1}(\phi, z + \Delta z)}}{\partial w(\phi)} = -\frac{1}{2} \cdot \frac{\Delta t_p}{\Delta\phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{r^2}{(r + \Delta z)^2} \cdot \frac{\cos(\phi + \Delta\phi)}{\cos(\phi)} \cdot \frac{v(\phi + \Delta\phi, z)}{r} \quad (59)$$

$$\frac{\partial \theta_{v_{i+1}(\phi, z)}}{\partial w(\phi, z)} = -\left(\frac{\Delta t_p}{\Delta z} \right)^2 \cdot \left(w(\phi, z) + \frac{w(\phi, z - \Delta z)}{2} \right) \quad (60)$$

$$\frac{\partial \theta_{v_{i+1}(\phi, z + \Delta z)}}{\partial w(\phi, z)} = -\left(\frac{\Delta t_p}{\Delta z} \right)^2 \cdot \frac{w(\phi, z + \Delta z)}{2} \quad (61)$$

$$\frac{\partial \theta_{v_{i+1}(\phi, z)}}{\partial w(\phi, z)} = \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta z} \cdot \left(1 + w(\phi, z - \Delta z) \cdot \frac{\Delta t_p}{\Delta z} \right) \quad (62)$$

$$\frac{\partial \theta_{v_{i+1}(\phi, z + \Delta z)}}{\partial w(\phi, z)} = \frac{1}{2} \cdot \left(\frac{\Delta t_p}{\Delta z} \right)^2 \cdot w(\phi, z + \Delta z) \quad (63)$$

$$\frac{\partial \theta_{v_{i+1}(\phi, z)}}{\partial w(\phi, z)} = -\frac{1}{2} \cdot \frac{\Delta t_p}{\Delta z} \cdot \left(1 - 2\frac{\Delta t_p}{\Delta z} \cdot w(\phi, z) \right) \quad (64)$$

4. Residual:



Trace gas contributions
predicted from initial values
measured at time t_0



Trace gas
distributions
measured at time t



DIFFERENCE

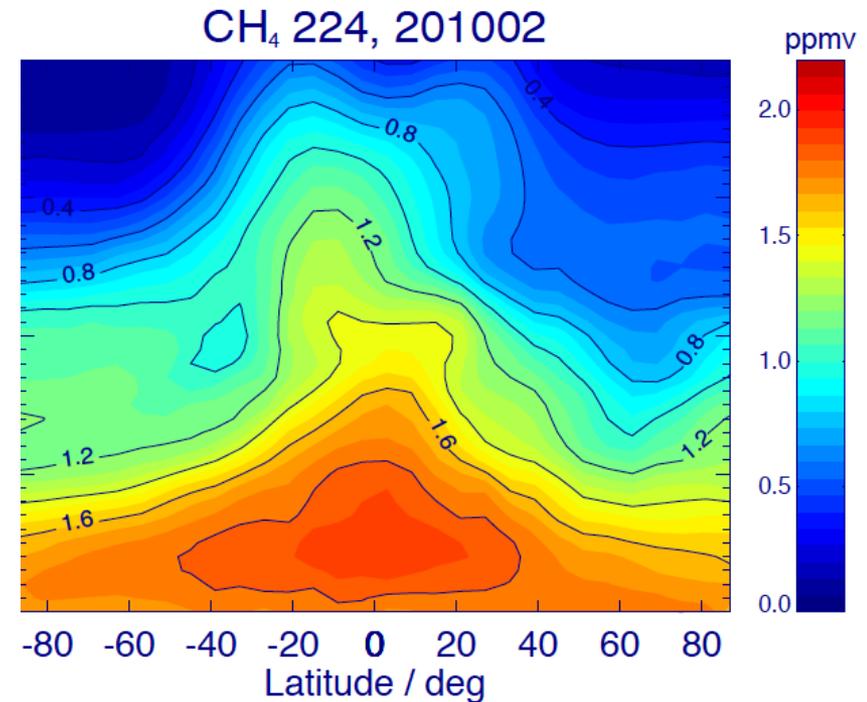
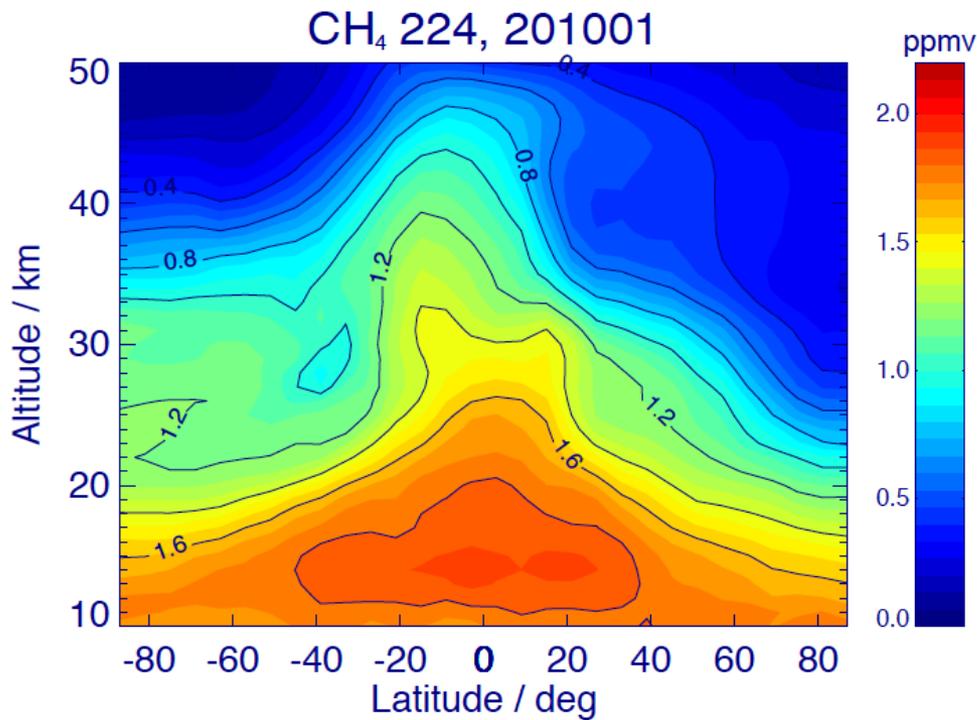
- Degradation of the spatial resolution of the kinetic fields works against the ill-posedness of the inverse problem. We maximize:

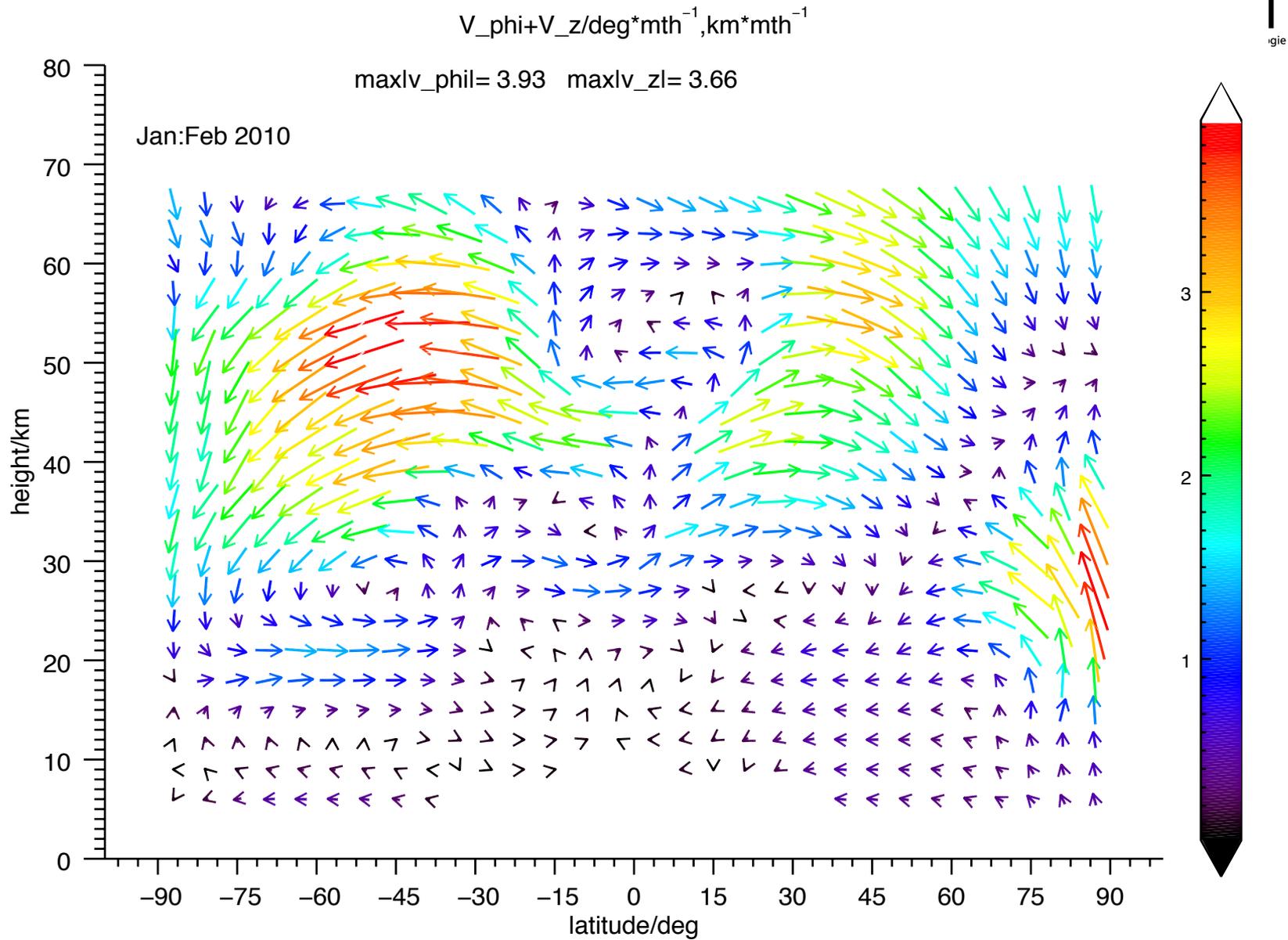
- pdf = $c_1 \exp[-\frac{1}{2}(y-F(x))^T S_y^{-1}(y-F(x))]$ $c_2 \exp[-\frac{1}{2}(x-x_a)^T R(x-x_a)]$
 = $c_3 \exp[-\frac{1}{2}(y-F(x))^T S_y^{-1}(y-F(x)) + (x-x_a)^T R(x-x_a)]$

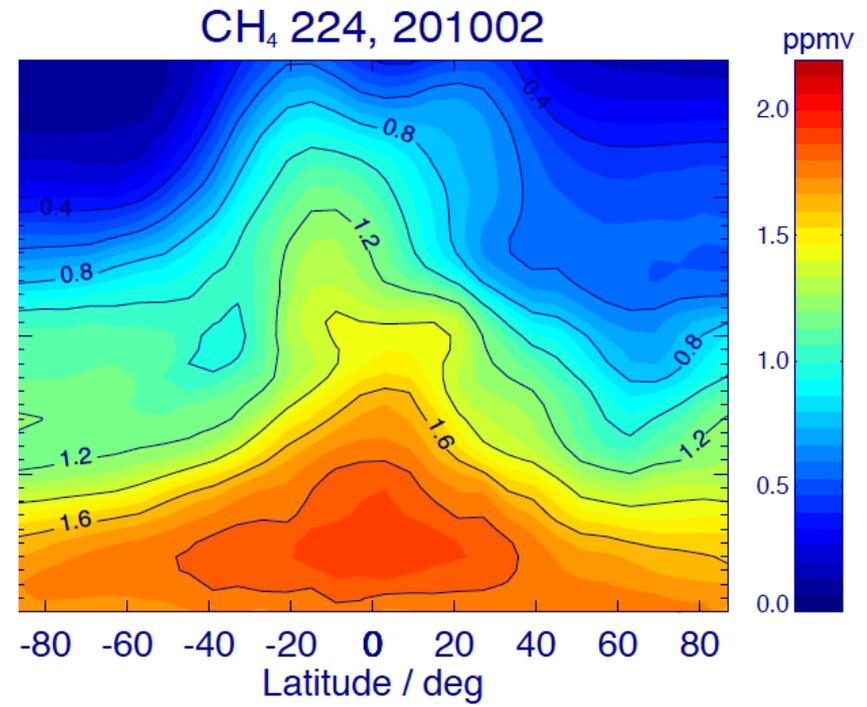
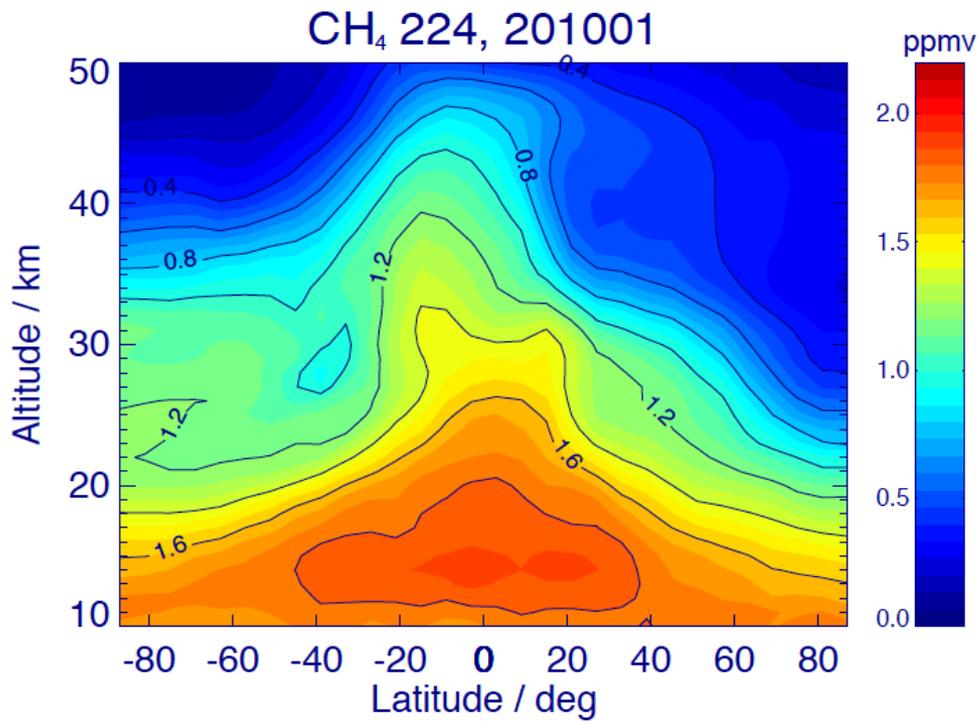
- $x_{\text{estimated}} = x_a + (K^T S_y^{-1} K + R)^{-1} K^T S_y^{-1}(y-F(x_a))$

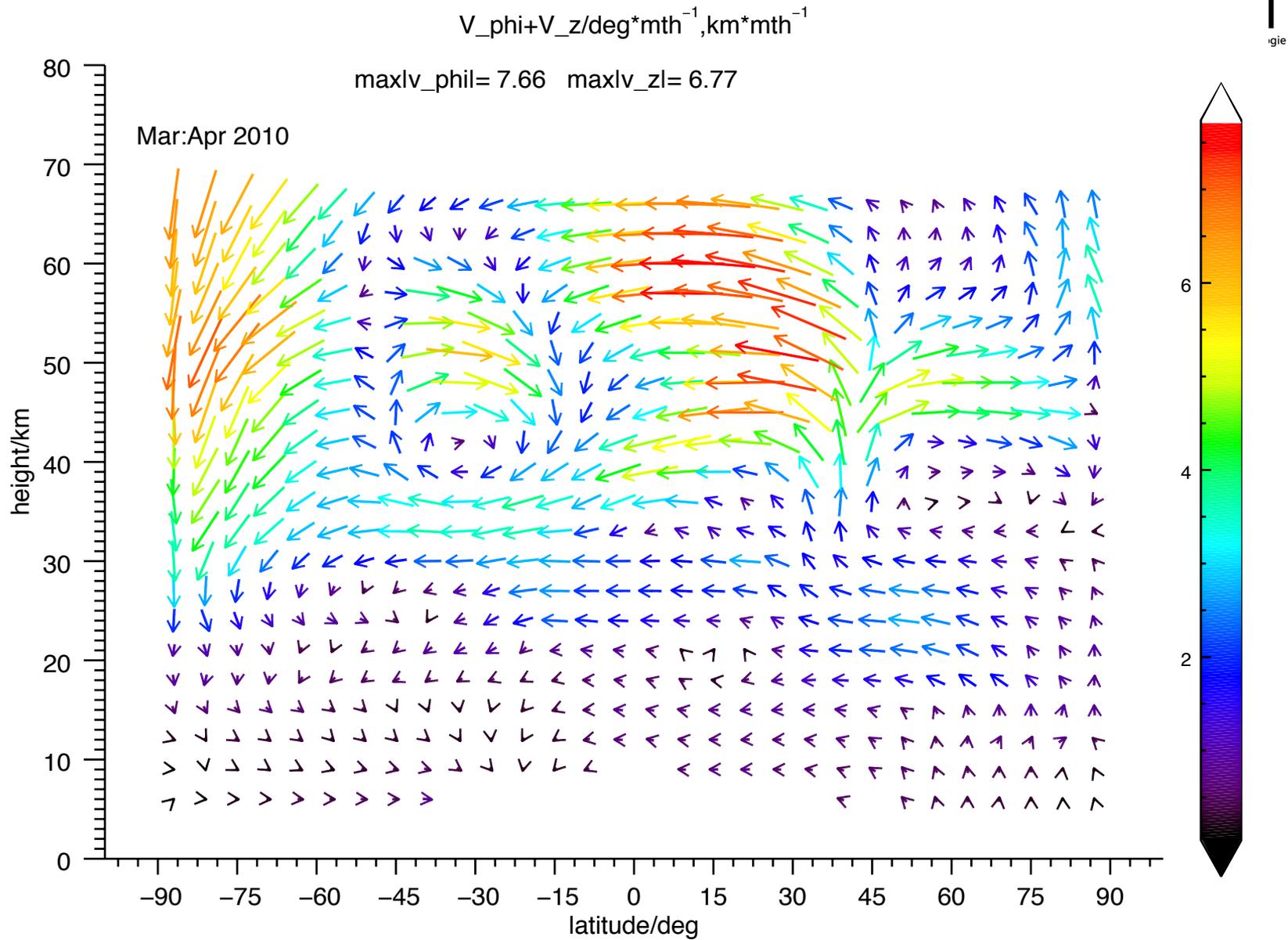
- These $x_{\text{estimated}}$ values are the kinematic variables obtained from this kind of tracer analysis.

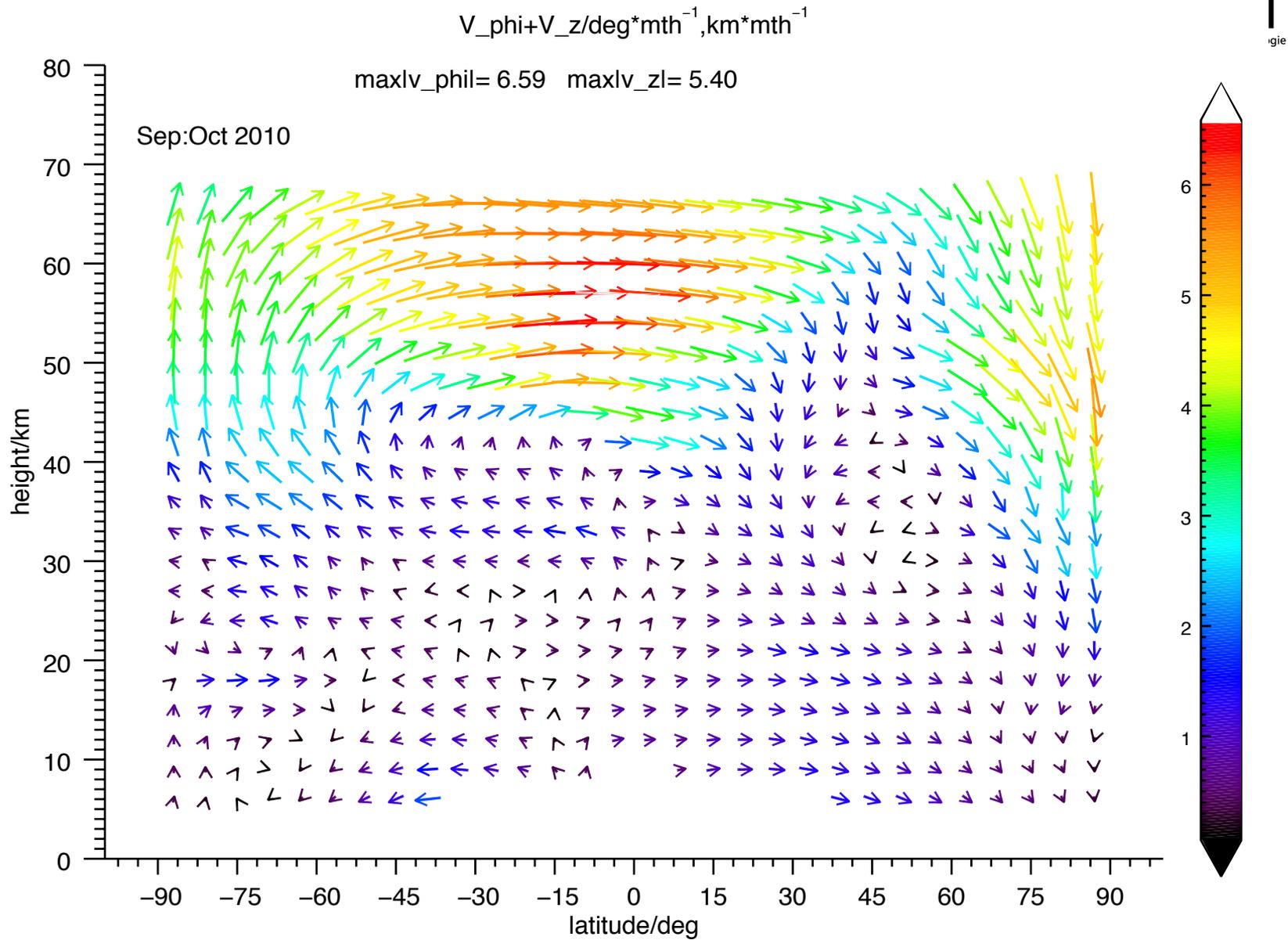
Test case: p, CH₄, N₂O, CFC-12, and SF₆



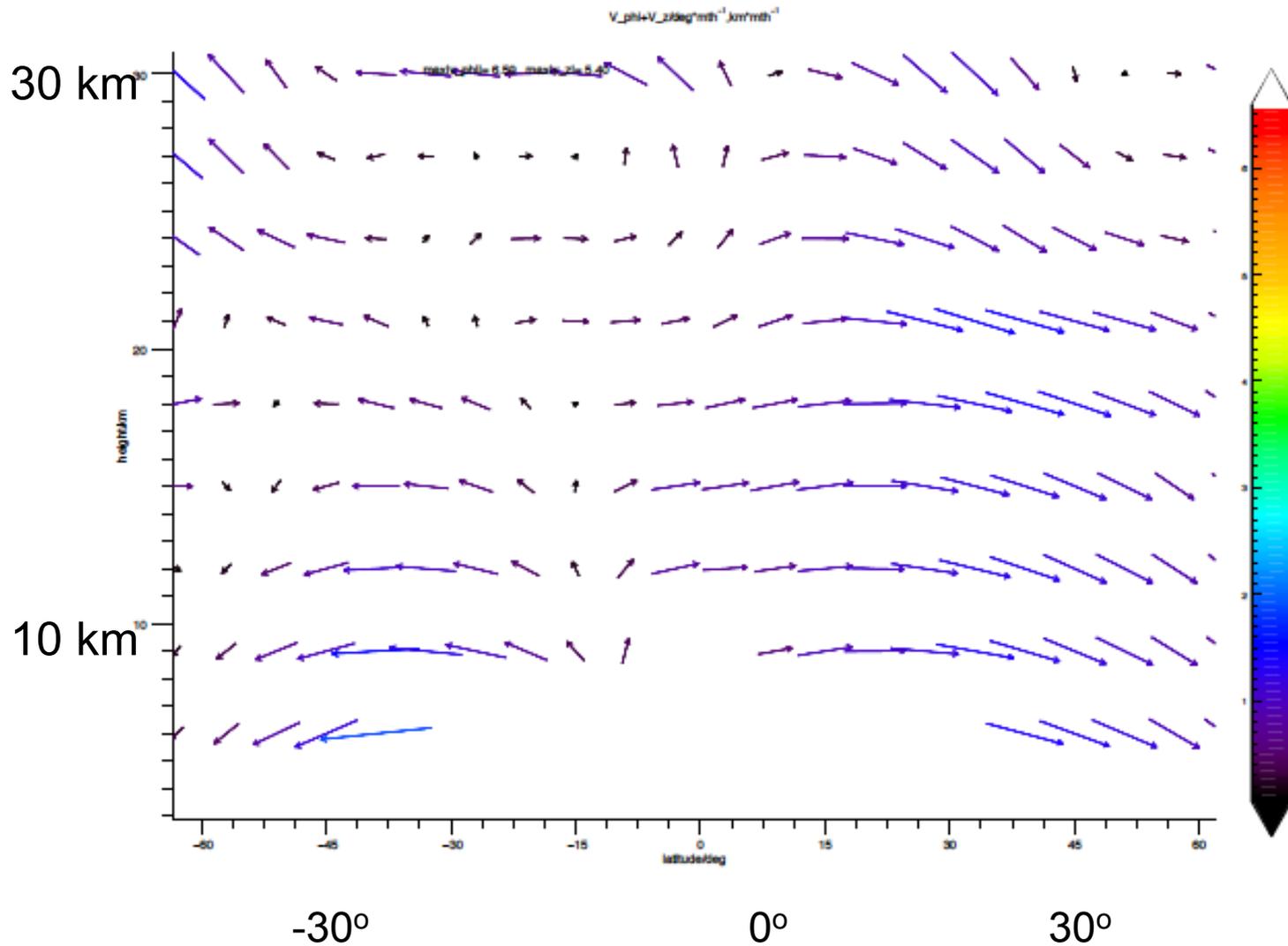


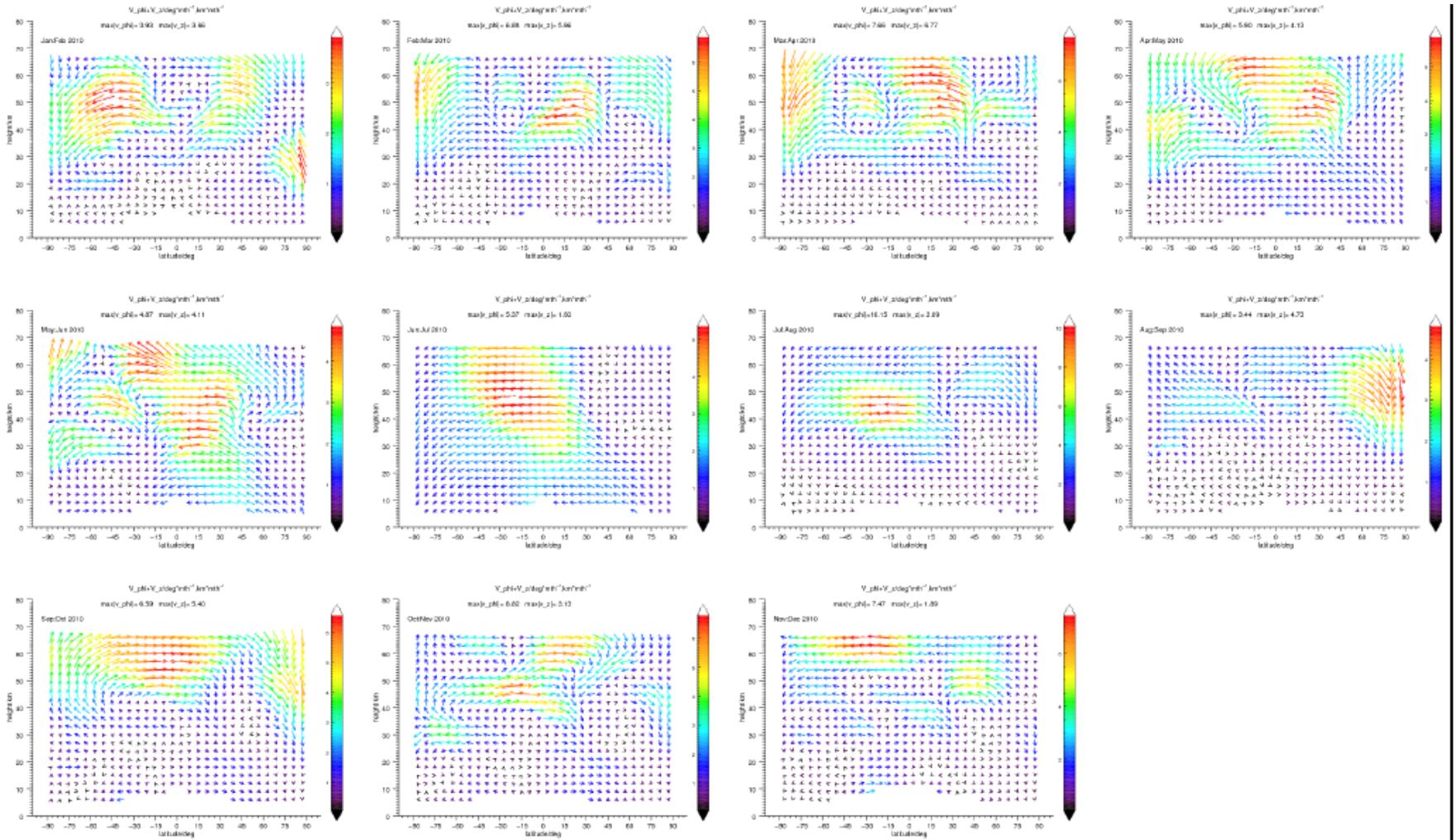






September – October 2010. Detail





Next steps

- Include more species;
- Consider sinks (photochemical, OH, O¹D);
- Tune constraints to best distinguish v-terms from K-terms;
- Analyze entire MIPAS period to validate if the scheme reproduces known events;
- Analyze MIPAS MA and UA measurements;
- Scientific analysis:
 - Brewer Dobson circulation (BDCHANGE and follow-up);
 - Overturning circulation (MESOENERGY);
 - Stratospheric – mesospheric exchange, subsidence of UA air etc (MESOENERGY);
 - Stratospheric – tropospheric exchange; (MESOENERGY)
 - ...
- Include other satellite missions;

$$\frac{\partial v_{mr_{i+1}}(\phi, z)}{\partial v(\phi, z)} = - \left(\frac{\Delta t_p}{\Delta \phi} \right)^2 \cdot \frac{1}{r^2} \cdot \left(v(\phi, z) + \frac{v(\phi - \Delta \phi, z)}{2} \right) \quad (46)$$

$$\frac{\partial \frac{\partial v_{mr_{i+1}}(\phi + \Delta \phi, z)}{\partial v_{mr_i}(\phi + \Delta \phi, z)}}{\partial v(\phi, z)} = - \left(\frac{\Delta t_p}{\Delta \phi} \right)^2 \cdot \frac{v(\phi + \Delta \phi, z)}{2r^2} \quad (47)$$

$$\frac{\partial \frac{\partial v_{mr_{i+1}}(\phi, z)}{\partial v_{mr_i}(\phi - \Delta \phi, z)}}{\partial v(\phi, z)} = \frac{1}{2r} \cdot \frac{\Delta t_p}{\Delta \phi} \cdot \left(1 + \frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi - \Delta \phi, z)}{r} \right) \quad (48)$$

$$\frac{\partial \frac{\partial v_{mr_{i+1}}(\phi + \Delta \phi, z)}{\partial v_{mr_i}(\phi, z)}}{\partial v(\phi, z)} = \frac{1}{2r} \cdot \left(\frac{\Delta t_p}{\Delta \phi} \right)^2 \cdot \frac{v(\phi + \Delta \phi, z)}{r} \quad (49)$$

$$\frac{\partial \frac{\partial v_{mr_{i+1}}(\phi, z)}{\partial v_{mr_i}(\phi + \Delta \phi, z)}}{\partial v(\phi, z)} = - \frac{\Delta t_p}{\Delta \phi} \cdot \frac{1}{r} \cdot \left(\frac{1}{2} - \frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi, z)}{r} \right) \quad (50)$$

Entries not mentioned here are zero. Entries relevant to w are:

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi, z)}{\partial \rho_i(\phi, z)}}{\partial w(\phi, z)} = \frac{\Delta t_p}{r} \cdot \frac{\Delta t_p}{r} \cdot v(\phi, z)$$

$$\frac{\partial w(\phi, z)}{\partial w(\phi, z)} = \frac{1}{2} \cdot \left(\frac{\Delta t_p}{\Delta z} \right)^2 \cdot \frac{(r + \Delta z)^2}{r^2} \cdot w(\phi, z + \Delta z) \quad (50)$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi, z - \Delta z)}{\partial \rho_i(\phi, z)}}{\partial w(\phi, z)} = \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{r^2}{(r - \Delta z)^2} \cdot \left(-1 + \frac{\Delta t_p}{\Delta \phi} \cdot \frac{v(\phi, z - \Delta z)}{r - \Delta z} + \frac{\Delta t_p}{\Delta z} \cdot w(\phi, z - \Delta z) \right) \quad (57)$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi + \Delta \phi, z - \Delta z)}{\partial \rho_i(\phi, z)}}{\partial w(\phi, z)} = - \frac{\Delta t_p}{2} \cdot \frac{\Delta t_p}{\Delta \phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{1}{r^2} \cdot \frac{\cos(\phi + \Delta \phi)}{\cos(\phi)} \cdot \frac{1}{(r - \Delta z)^2} \cdot \frac{1}{r - \Delta z} \quad (58)$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi, z + \Delta z)}{\partial \rho_i(\phi + \Delta \phi, z)}}{\partial w(\phi)} = \frac{1}{2} \cdot \frac{\Delta t_p}{\Delta \phi} \cdot \frac{\Delta t_p}{\Delta z} \cdot \frac{r^2}{(r + \Delta z)^2} \cdot \frac{\cos(\phi + \Delta \phi)}{\cos(\phi)} \cdot \frac{v(\phi + \Delta \phi, z)}{r} \quad (59)$$

$$\frac{\partial \frac{\partial v_{mr_{i+1}}(\phi, z)}{\partial v_{mr_i}(\phi, z)}}{\partial w(\phi, z)} = \frac{1}{2} \cdot \left(\frac{\Delta t_p}{\Delta z} \right)^2 \cdot \frac{1}{r} \cdot w(\phi, z - \Delta z) \quad (60)$$

THANK YOU
and sorry for the maths