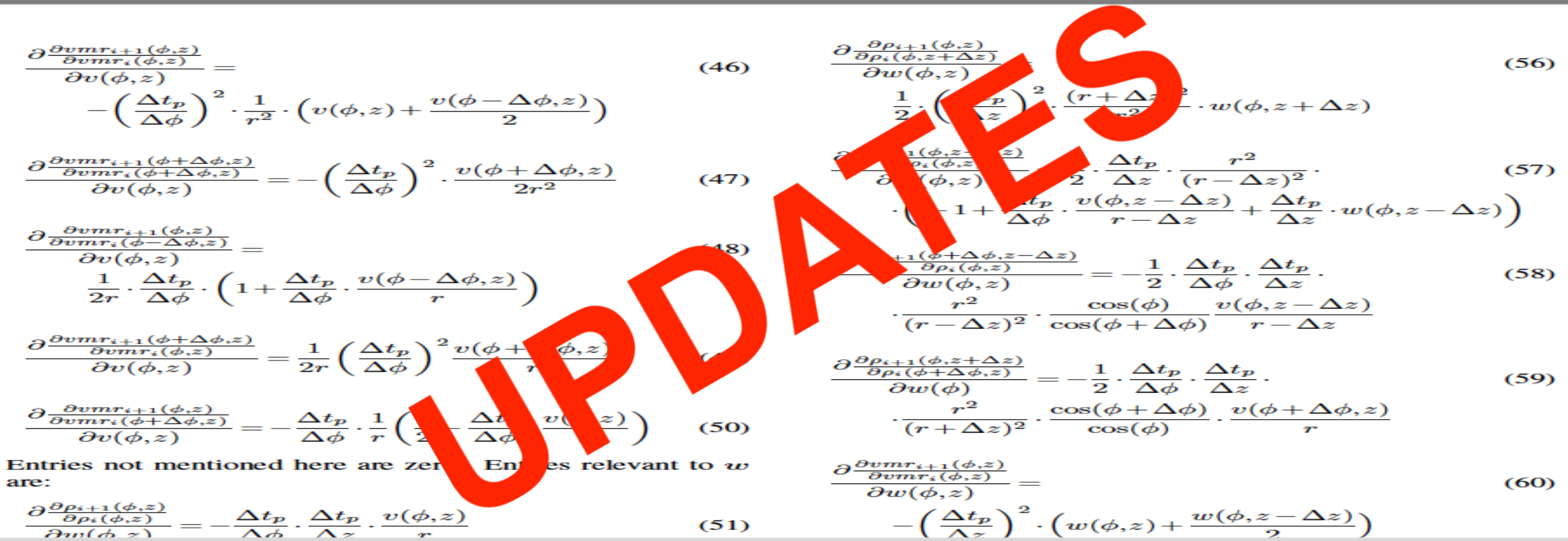
Karlsruher Institut für Technologie

# Another Approach to the Brewer-Dobson-Circulation: the direct inversion of the Continuity Equation:


## Thomas von Clarmann and Udo Grabowski


## ROMIC-2016


IMK-ASF-SAT

# Before telling you what's new…

ANCISTRUS SPEC.

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Analysis of the Circulation of the Stratosphere using spectroscopic measurements

# ...let's recap the old stuff

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Goal: Empirical analysis of the BDC using tracer measurements

- The age of air can be understood as an "integral over the stratospheric residence time of the air parcel". Roughly speaking, the intensity of the circulation is the quotient of the displacement of the air parcel during its stratospheric life and its stratospheric residence time.

- If we find an approach which is more differential, we can avoid the problems mentioned before: The strength of the circulation then is the quotient of a short-time displacement and the related time-step.

- The continuity equation helps to achieve this!

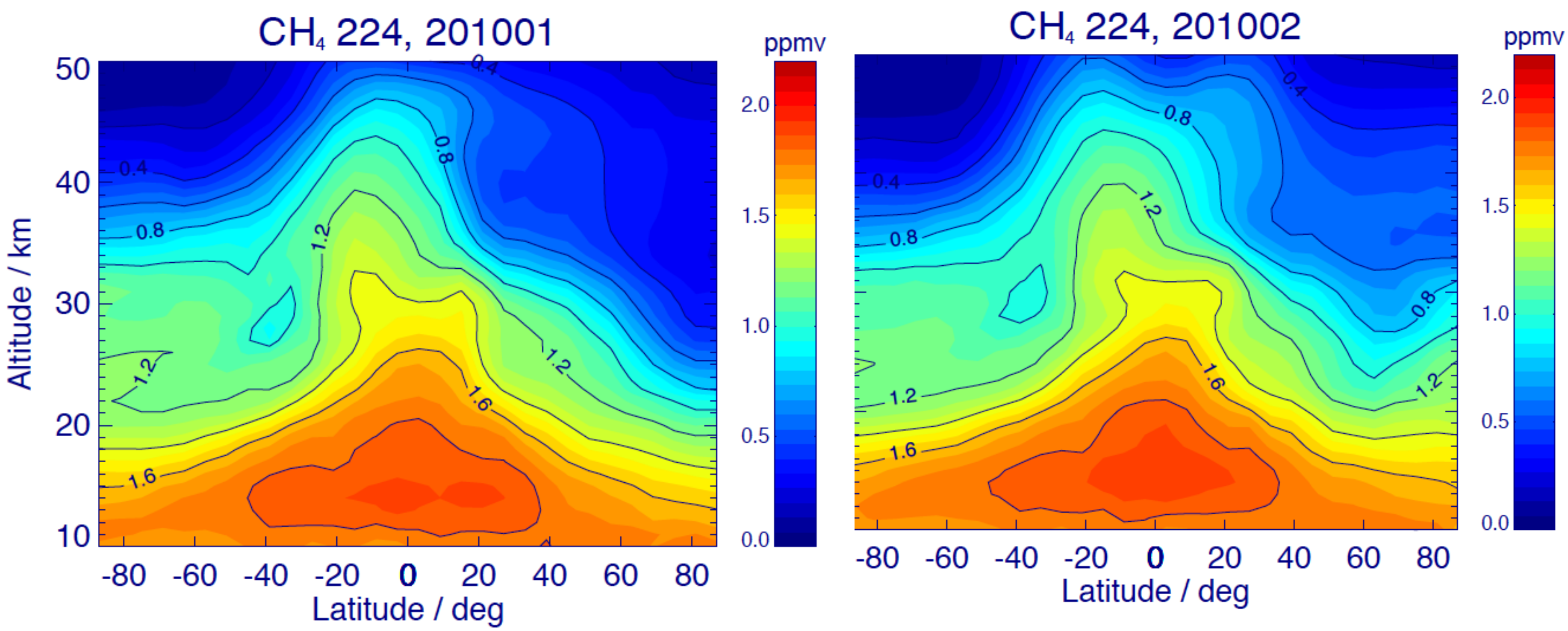
# 2-D continuity equation

- Usual transport calculations (forward modelling of transport):
  (vmr, density) = continuity equation ($v,w,K_{\phi},K_z$; $vmr_0$,$density_0$)

- What we do (inverse modeling of circulation):
  ($v,w,K_{\phi},K_z$) = continuity equation $^{-1}$ (vmr,density,$vmr_0$,$density_0$)

For us, tracer analysis is the inverse solution of the continuity equation.

# We start with monthly zonal mean vmr distributions (in our case: MIPAS)



# The major steps

1. Predict the later state for a given initial state by the `forward solution' of the continuity equation for assumed kinematic variables (transport calculation).
2. Estimate the uncertainties of the predicted later state (error estimation).
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# Step 1: Transport modeling:

- We integrate the tendencies:
  new state = old state +δstate/δkinematic * kinematic

- Simple Euler integration is too diffusive.

- State of the art integration (e.g. Prather) does not allow a closed form solution for the Jacobians.

- Compromise: MacCormack-integration: a predictor-corrector method.

$$c^*_{i+1}(x,y) = c_i(x,y) - \frac{\Delta t_p}{\Delta x}(e_i(x+\Delta x,y) - e_i(x,y)) - \frac{\Delta t_p}{\Delta y}(f_i(x,y+\Delta y) - f_i(x,y)).$$

$$c_{i+1}(x,y) = \frac{1}{2}\Big[c_i(x,y) + c^*_{i+1}(x,y) - \frac{\Delta t_p}{\Delta x}\left(e(c^*_{i+1},x,y) - e(c^*_{i+1},x-\Delta x,y)\right) - \frac{\Delta t_p}{\Delta y}\left(f(c^*_{i+1},x,y) - f(c^*_{i+1},x,y-\Delta y)\right)\Big]$$

locations. The sensitivities of the densities of the first predictive step with respect to the initial densities at the same latitude and altitude are

$$\frac{\partial\rho_{i+1}(\phi,z)}{\partial\rho_i(\phi,z)} = \frac{1}{2}\Bigg[2-\frac{\Delta t_p}{\Delta\phi}\Big[\frac{v(\phi,z)}{r}\Big(\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi,z)}{r}+\frac{\Delta t_p}{\Delta z}w(\phi,z)\Big) + \frac{\Delta t_p}{\Delta\phi}\frac{v(\phi-\Delta\phi,z)v(\phi,z)}{r^2}\Big] - \frac{\Delta t_p}{\Delta z}\Big[w(\phi,z)\Big(\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi,z)}{r}+\frac{\Delta t_p}{\Delta z}w(\phi,z)\Big)+\frac{\Delta t_p}{\Delta z}w(\phi,z-\Delta z)w(\phi,z)\Big]\Bigg] \quad (15)$$

We further differenciate predicted air densities with respect to air densities at the adjacent southern latitude but the same altitude.

$$\frac{\partial\rho_{i+1}(\phi,z)}{\partial\rho_i(\phi-\Delta\phi,z)} = \frac{1}{2}\Big[\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi-\Delta\phi,z)}{r}\cdot\frac{\cos(\phi-\Delta\phi)}{\cos(\phi)}\Big(1+\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi-\Delta\phi,z)}{r}+\frac{\Delta t_p}{\Delta z}w(\phi-\Delta\phi,z)\Big)\Big] \quad (16)$$

The derivative of the predicted air densities with respect to air densities at the adjacent northern latitude but the same altitude is

$$\frac{\partial\rho_{i+1}(\phi,z)}{\partial\rho_i(\phi-\Delta\phi,r+\Delta z)} = -\frac{1}{2}\Big[\frac{v(\phi-\Delta\phi,z)}{r}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\frac{\Delta t_p}{\Delta z}\cdot\frac{(r+\Delta z)^2}{r^2}\cdot\frac{\cos(\phi-\Delta\phi)}{\cos(\phi)}w(\phi-\Delta\phi,r+\Delta z)\Big] \quad (20)$$

and vice versa

$$\frac{\partial\rho_{i+1}(\phi,z)}{\partial\rho_i(\phi+\Delta\phi,r-\Delta z)} = -\frac{1}{2}\Big[w(\phi,r-\Delta z)\frac{\Delta t_p}{\Delta z}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\frac{(r-\Delta z)^2}{r^2}\cdot\frac{\cos(\phi+\Delta\phi)}{\cos(\phi)}\cdot\frac{v(\phi+\Delta\phi,z-\Delta z)}{r-\Delta z}\Big]$$

where $i$ is the index of the time increment, … and $z\pm\Delta z$ refer to the adjacent model gri… and altitude, respectively.

For mixing ratios, the respective derivati…

$$\frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi,z)} = 1 - \left(\frac{\Delta t_p}{\Delta\phi}\right)^2\cdot\frac{v(\phi,z)}{r^2}\cdot\frac{1}{2}[v(\phi,z)+v(\phi-\Delta \ldots$$

…



$$\frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi+\Delta\phi,z)} = -\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi,z)}{2r}\Big(1-\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi,z)}{r}\Big)+\frac{\Delta t_p}{2r^2(\Delta\phi)^2\cos(\phi)}\cdot\Big(K_\phi(\phi,z)+K_\phi(\phi+\Delta\phi,z)\Big)\cos(\phi+\frac{\Delta\phi}{2}); \quad (23)$$

$$\frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi-\Delta\phi,z)} = \frac{v(\phi,z)}{2r}\cdot\frac{\Delta t_p}{\Delta\phi}\Big(1+\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi-\Delta\phi,z)}{r}\Big)+\frac{\Delta t_p}{2r^2(\Delta\phi)^2\cos(\phi)}\cdot\Big(K_\phi(\phi,z)+K_\phi(\phi-\Delta\phi,z)\Big)\cos(\phi-\frac{\Delta\phi}{2}); \quad (24)$$

$$\frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi,z+\Delta z)} = -\frac{1}{2}\cdot w(\phi,z)\cdot\frac{\Delta t_p}{\Delta z}\Big(1-\frac{(\Delta t_p)}{(\Delta z)}w(\phi,z)\Big)+\frac{\Delta t_p}{2r^2(\Delta z)^2}(r+\frac{\Delta z}{2})^2\Big(K_z(\phi,z)+K_z(\phi,z+\Delta z)\Big); \quad (25)$$

$$\frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi,z-\Delta z)} = \frac{\Delta t_p}{\Delta z}\cdot\frac{1}{2}\cdot w(\phi,z)\Big(1+w(\phi,z-\Delta z)\frac{\Delta t_p}{\Delta z}\Big)+\frac{\Delta t_p}{2r^2(\Delta z)^2}(r-\frac{\Delta z}{2})^2\Big(K_z(\phi,z)+K_z(\phi,z-\Delta z)\Big); \quad (26)$$

These derivatives are simplifications in a sense that they do not consider the full chemical Jacobian but assume instead that the source strength depends on no other concentration than the actual concentration of the same species. For the typical long-lived so-called tropospheric source gases considered here, like $SF_6$ or CFCs, this assumption is appropriate.

With these expressions, the prediction of air density and volume mixing ratio can be rewritten in matrix notation for a …

where

- $\mathbf{D}_{\rho;i}$ is the $L_0\times L_0$ Jacobian matrix of air density for time increment $i$, i.e. the sensitivities of the prediction with respect to the initial state, $\frac{\partial c_{i+1,m}}{\partial c_{i,n}}$, here $m$ and $n$ run over the model gridpoints
- $\mathbf{I}_K$ is $K_0\times K_0$ identity;
- $\mathbf{0}$ are zero submatrices of the required dimensions;
- $\mathbf{W}_i$ is a $K_0\times 2K_0$-dimensional interpolati… matrix;
- $\mathbf{D}_{\rho;\mathrm{J}}$ is an $J_0\times L_0$ Jacobian containing the partial derivatives $\partial\rho_{i+1;j}/\partial\rho_{i;l}$;
- $\rho_{I;k=1,K_0}$ is the $K_0$-dimensional vector of air densities in the border region after the final timestep, i.e. for the time of the n… measurement;
- $\rho_{i;k=1,K_0}$ is the $K_0$-dimensional vector of air densities in the border region at the current timestep as resulting from interpolation in time;
- $\rho_{i;j=K_0+1,L_0}$ is the $K_0$-dimensional vector of air densities in the nominal region at the current timestep as resulting from integration according to the MacCorma… scheme as described above.

Since the source term depends on air density, the integra… in matrix notation for vmr requires simultaneous treatm… of vmr and air density and we get, using notation accor… with air density:

$$\begin{pmatrix}\rho_{i+1}\\ vmr_{i+1}\end{pmatrix} = \begin{pmatrix}\rho_{l=1}\\ \vdots\\ \rho_{L_0}\\ vmr_{g;l=1}\\ \vdots\\ vmr_{g;\sum L_g}\end{pmatrix} = \mathbf{D}_i\begin{pmatrix}\rho_i\\ vmr_{g;i}\end{pmatrix} = \ldots$$

$$= \begin{pmatrix}\mathbf{D}_{\rho;i} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_K & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{W}_i & & \mathbf{0}\\ & & \mathbf{D}_{g;\mathrm{J}} & \end{pmatrix}\begin{pmatrix}\rho_{i,l=1,L_0}\\ vmr_{g;I,k=1,K_g}\\ vmr_{g;i,k=1,K_g}\\ vmr_{g;i,j=K_g+1,L_g}\end{pmatrix}$$

where $\mathbf{D}_i$ is the total Jacobian with respect to air densi… and all involved gas mixing ratios. Note that

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# Step 2: Error Propagation

It is convenient to write the transport problem in matrix notation:

Later state = D * initial state = $D_n$ * … *$D_2$*$D_1$*initial state;

Each $D_{i,i=1,n}$ refers to one 'micro-timestep';

In this case, the prediction error $S_I$ is calculated from the uncertainty of the initial state $S_0$ via generalized Gaussian error propagation as

$$S_I = \Pi D_n S_0 (\Pi D_n)^T ;$$

The "only" problem is to get the $D_n$ matrices ☺

have fun……

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efficiently computed using the following recursive scheme, where $\tilde{f}_{l,i}$ is the respective column of the Jacobian after micro timestep $i$:

$$\tilde{f}_{n,i} = \mathrm{D}_i f_{n,i-1} + \frac{\partial \mathrm{D}_i}{\partial q_n}\left(\prod_{k=i-1}^{1} \mathrm{D}_k\right)\tilde{x}_0 \tag{36}$$

With the argument of D specifying the column of the D-matrix such that $D_{c,i}(\phi,z)$ relates $\rho_{i+1}(\phi,z)$ to $\rho_i(\phi,z)$, $D_{\rho,i}(\phi \pm \Delta\phi,z)$ relates $\rho_{i+1}(\phi,z)$ to $\rho_i(\phi \pm \Delta\phi,z)$, and $D_{\rho,i}(\phi,z \pm \Delta z)$ relates $\rho_{i+1}(\phi,z)$ to $\rho_i(\phi,z \pm \Delta z)$, and for vmr accordingly, the entries of $\mathrm{D}_i$ relevant to $v$ are:

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi,z)}{\partial \rho_i(\phi,z)}}{\partial v(\phi,z)} = -\frac{\Delta t_p}{2\Delta\phi} \cdot \left(\frac{\Delta t_p}{\Delta\phi}\cdot\frac{2v(\phi,z)+v(\phi-\Delta\phi,z)}{r^2} + 2\frac{\Delta t_p}{\Delta z}\cdot\frac{w(\phi,z)}{r}\right) \tag{37}$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi+\Delta\phi,z)}{\partial \rho_i(\phi+\Delta\phi,z)}}{\partial v(\phi,z)} = -\frac{1}{2}\cdot\left(\frac{\Delta t_p}{\Delta\phi}\right)^2\cdot\frac{v(\phi+\Delta\phi,z)}{r^2} \tag{38}$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi+\Delta\phi,z)}{\partial \rho_i(\phi,z)}}{\partial v(\phi\phi,z)} = \frac{1}{2r}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\frac{\cos(\phi)}{\cos(\phi+\Delta\phi)} \cdot \left(1+2\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi,z)}{r}+\frac{\Delta t_p}{\Delta z}w(\phi,z)\right) \tag{39}$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi,z)}{\partial \rho_i(\phi+\Delta\phi,z)}}{\partial v(\phi,z)} = \frac{1}{2}\cdot\left(\frac{\Delta t_p}{\Delta\phi}\right)^2\cdot\frac{v(\phi+\Delta\phi,z)}{r^2}\cdot\frac{\cos(\phi+\Delta\phi)}{\cos(\phi)} \tag{40}$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi-\Delta\phi,z)}{\partial \rho_i(\phi,z)}}{\partial v(\phi,z)} = \frac{1}{2r}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\frac{\cos(\phi)}{\cos(\phi-\Delta\phi)} \cdot \left(-1+\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi-\Delta\phi,z)}{r}+\frac{\Delta t_p}{\Delta z}w(\phi-\Delta\phi,z)\right) \tag{41}$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi,z+\Delta z)}{\partial \rho_i(\phi,z)}}{\partial v(\phi,z)} = \frac{1}{2}\cdot\frac{\Delta t_p}{\Delta z}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\frac{r^2}{(r+\Delta z)^2}\,\frac{w(\phi,z)}{r} \tag{42}$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi,z)}{\partial \rho_i(\phi,z+\Delta z)}}{\partial v(\phi,z)} = \frac{1}{2}\cdot\frac{\Delta t_p}{\Delta z}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\frac{(r+\Delta z)^2}{r^3}w(\phi,z+\Delta z) \tag{43}$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi+\Delta\phi,z)}{\partial \rho_i(\phi,z+\Delta z)}}{\partial v(\phi,z)} = -\frac{1}{2}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\frac{\Delta t_p}{\Delta z}\cdot \frac{(r+\Delta z)^2}{r^3}\cdot\frac{\cos(\phi)}{\cos(\phi+\Delta\phi)}w(\phi,z+\Delta z) \tag{44}$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi-\Delta\phi,z+\Delta z)}{\partial \rho_i(\phi,z)}}{\partial v(\phi,z)} = -\frac{1}{2}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\frac{\Delta t_p}{\Delta z}\cdot \frac{r}{(r+\Delta z)^2}\cdot\frac{\cos(\phi)}{\cos(\phi-\Delta\phi)}\cdot w(\phi-\Delta\phi,z) \tag{45}$$

$$\frac{\partial \frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi,z)}}{\partial v(\phi,z)} = -\left(\frac{\Delta t_p}{\Delta\phi}\right)^2\cdot\frac{1}{r^2}\cdot\left(v(\phi,z)+\frac{v(\phi-\Delta\phi,z)}{2}\right) \tag{46}$$

$$\frac{\partial \frac{\partial vmr_{i+1}(\phi+\Delta\phi,z)}{\partial vmr_i(\phi+\Delta\phi,z)}}{\partial v(\phi,z)} = -\left(\frac{\Delta t_p}{\Delta\phi}\right)^2\cdot\frac{v(\phi+\Delta\phi,z)}{2r^2} \tag{47}$$

$$\frac{\partial \frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi-\Delta\phi,z)}}{\partial v(\phi,z)} = \frac{1}{2r}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\left(1+\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi-\Delta\phi,z)}{r}\right) \tag{48}$$

$$\frac{\partial \frac{\partial vmr_{i+1}(\phi+\Delta\phi,z)}{\partial vmr_i(\phi,z)}}{\partial v(\phi,z)} = \frac{1}{2r}\left(\frac{\Delta t_p}{\Delta\phi}\right)^2\frac{v(\phi+\Delta\phi,z)}{r} \tag{49}$$

$$\frac{\partial \frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi+\Delta\phi,z)}}{\partial v(\phi,z)} = -\frac{\Delta t_p}{\Delta\phi}\cdot\frac{1}{r}\left(\frac{1}{2}-\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi,z)}{r}\right) \tag{50}$$

Entries not mentioned here are zero. Entries relevant to $w$ are:

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi,z)}{\partial \rho_i(\phi,z)}}{\partial w(\phi,z)} = -\frac{\Delta t_p}{\Delta\phi}\cdot\frac{\Delta t_p}{\Delta z}\cdot\frac{v(\phi,z)}{r} - \left(\frac{\Delta t_p}{\Delta z}\right)^2 w(\phi,z) - \frac{1}{2}\left(\frac{\Delta t_p}{\Delta z}\right)^2 w(\phi,z-\Delta z) \tag{51}$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi,z+\Delta z)}{\partial \rho_i(\phi,z+\Delta z)}}{\partial w(\phi,z)} = -\frac{1}{2}\left(\frac{\Delta t_p}{\Delta z}\right)^2 w(\phi,z+\Delta z) \tag{52}$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi,z)}{\partial \rho_i(\phi+\Delta\phi,z)}}{\partial w(\phi,z)} = \frac{1}{2}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\frac{\Delta t_p}{\Delta z}\cdot\frac{v(\phi+\Delta\phi,z)}{r}\cdot\frac{\cos(\phi+\Delta\phi)}{\cos(\phi)} \tag{53}$$

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$$\frac{\partial \frac{\partial \rho_{i+1}(\phi,z)}{\partial \rho_i(\phi,z+\Delta z)}}{\partial w(\phi,z)} = \frac{1}{2}\cdot\left(\frac{\Delta t_p}{\Delta z}\right)^2\cdot\frac{(r+\Delta z)^2}{r^2}\cdot w(\phi,z+\Delta z) \tag{56}$$

$$\frac{\partial \frac{\partial \rho_{i+1}(\phi,z-\Delta z)}{\partial \rho_i(\phi,z)}}{\partial w(\phi,z)} = \frac{1}{2}\cdot\frac{\Delta t_p}{\Delta z}\cdot\frac{r^2}{(r-\Delta z)^2}\cdot \left(-1+\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi,z-\Delta z)}{r-\Delta z}+\frac{\Delta t_p}{\Delta z}\cdot w(\phi,z-\Delta z)\right) \tag{57}$$

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$$\frac{\partial \frac{\partial \rho_{i+1}(\phi,z+\Delta z)}{\partial \rho_i(\phi+\Delta\phi,z)}}{\partial w(\phi)} = -\frac{1}{2}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\frac{\Delta t_p}{\Delta z}\cdot \frac{r^2}{(r+\Delta z)^2}\cdot\frac{\cos(\phi+\Delta\phi)}{\cos(\phi)}\cdot\frac{v(\phi+\Delta\phi,z)}{r} \tag{59}$$

$$\frac{\partial \frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi,z)}}{\partial w(\phi,z)} = -\left(\frac{\Delta t_p}{\Delta z}\right)^2\cdot\left(w(\phi,z)+\frac{w(\phi,z-\Delta z)}{2}\right) \tag{60}$$

$$\frac{\partial \frac{\partial vmr_{i+1}(\phi,z+\Delta z)}{\partial vmr_i(\phi,z+\Delta z)}}{\partial w(\phi,z)} = -\left(\frac{\Delta t_p}{\Delta z}\right)^2\cdot\frac{w(\phi,z+\Delta z)}{2} \tag{61}$$

$$\frac{\partial \frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi,z-\Delta z)}}{\partial w(\phi,z)} = \frac{1}{2}\cdot\frac{\Delta t_p}{\Delta z}\cdot\left(1+w(\phi,z-\Delta z)\cdot\frac{\Delta t_p}{\Delta z}\right) \tag{62}$$

$$\frac{\partial \frac{\partial vmr_{i+1}(\phi,z+\Delta z)}{\partial vmr_i(\phi,z)}}{\partial w(\phi,z)} = \frac{1}{2}\cdot\left(\frac{\Delta t_p}{\Delta z}\right)^2\cdot w(\phi,z+\Delta z) \tag{63}$$

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# 4. Resiudual:

Trace gas contributions predicted from initial values measured at time $t_0$

Trace gas distributions measured at time t

DIFFERENCE

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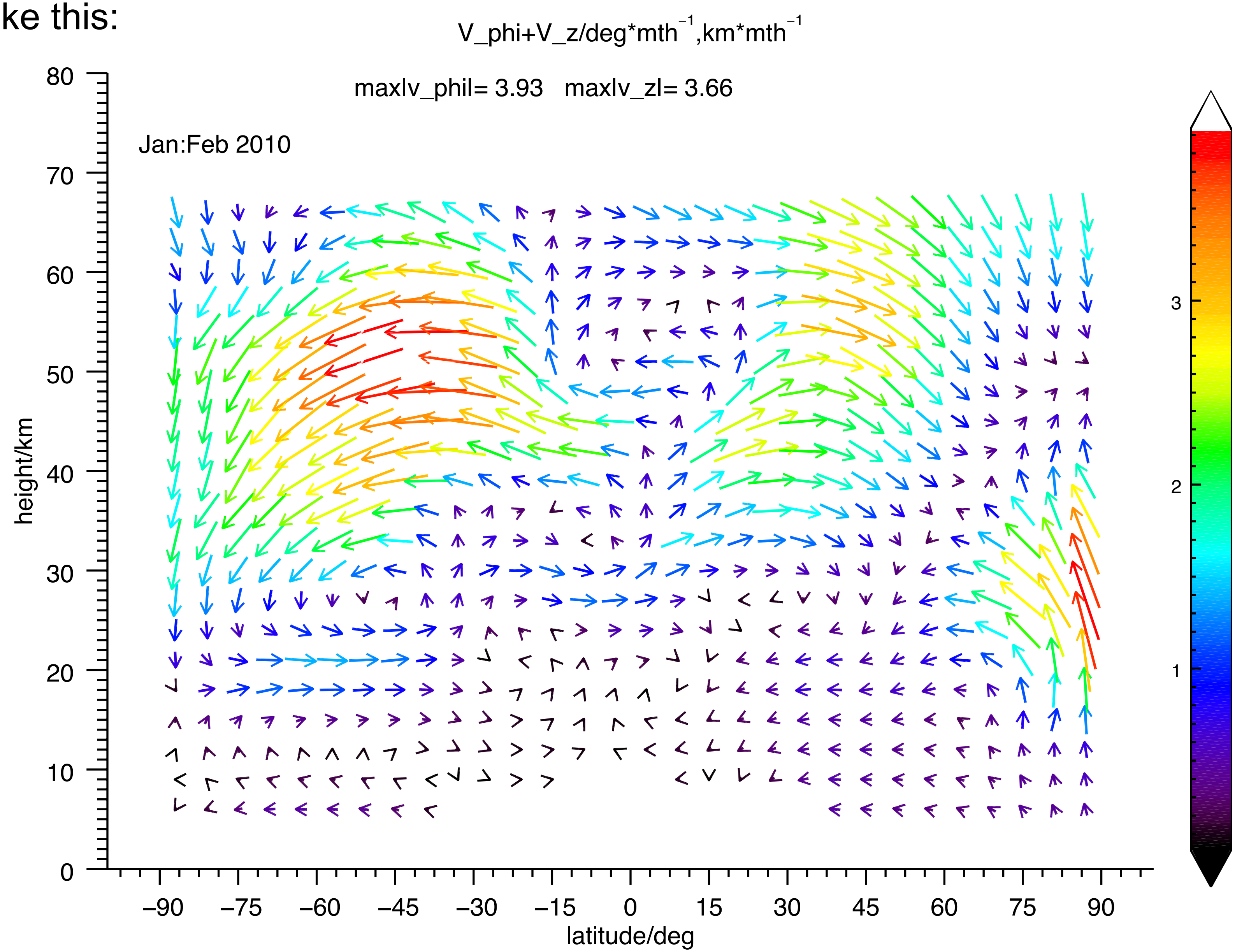
like this:



V_phi+V_z/deg*mth$^{-1}$,km*mth$^{-1}$
maxlv_phil= 3.93 maxlv_zl= 3.66
Jan:Feb 2010
height/km
latitude/deg
80
70
60
50
40
30
20
10
0
−90 −75 −60 −45 −30 −15 0 15 30 45 60 75 90
3
2
1

# This was all old stuff.

# What’s new?

# What's new?

- Sinks of all relevant gases have been implemented (photochemistry, OH chemistry, $O^1D$ chemistry)
- Sources of $H_2O$ and CO have been implemented
- And finally we have tried to understand a little bit better what we are doing…

…which led us to the question where the information comes from.

Karlsruher Institut für Technologie

# Two pathways of information:

1. Tracing transported patterns; can best be analyzed by setting the sources and sinks to zero.
2. Balance between sinks and transport; can best be analyzed by analyzing stationary fields.

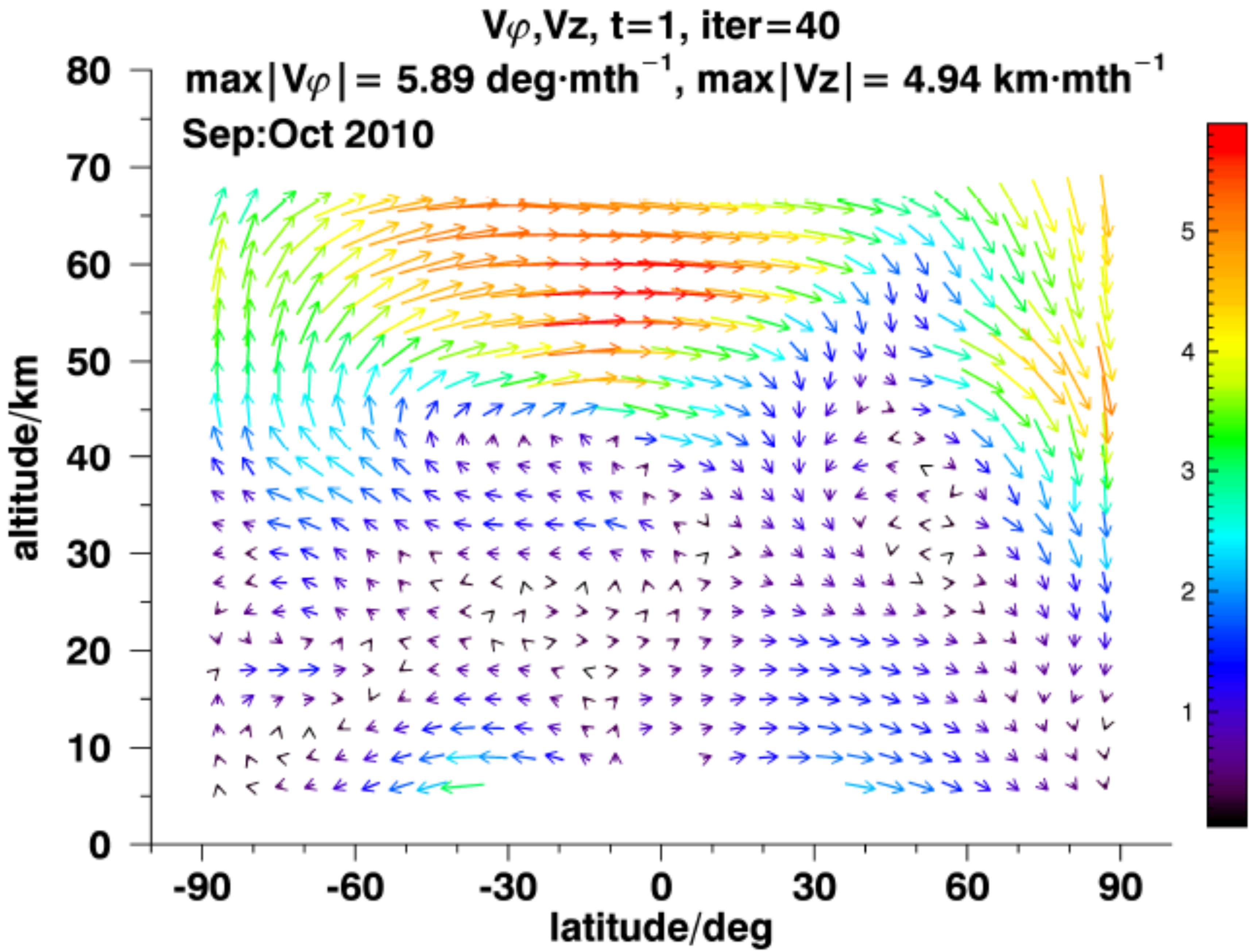
- The realistic result contains a component of both processes.

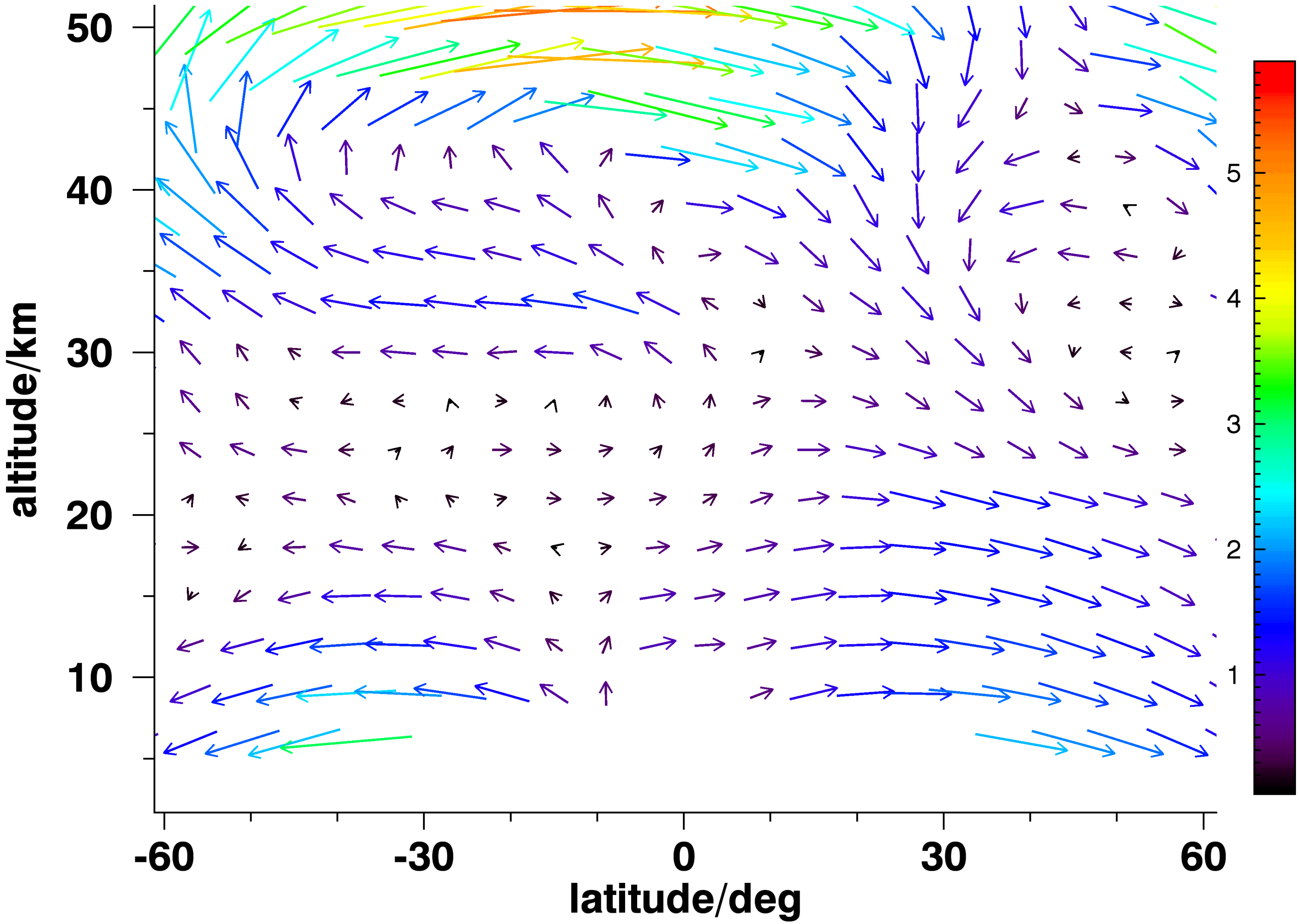
# Test case 1:

# Sep-Oct 2010 vmr fields from MIPAS

# No sinks/sources – tracing of patterns (local vmr changes) only

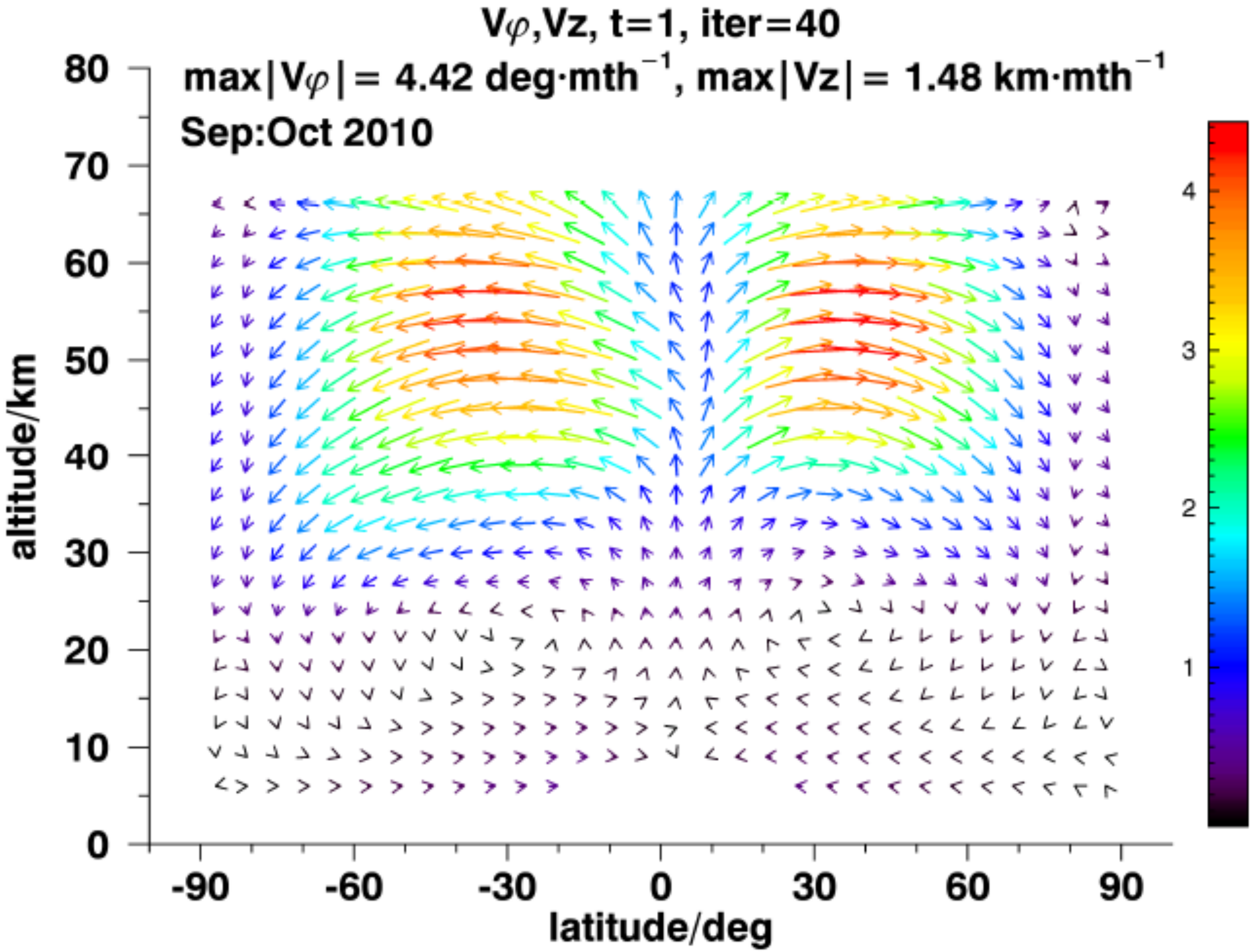
Gases used: CFC-12, $CH_4$, $N_2O$, $SF_6$


Vφ,Vz, t=1, iter=40
max|Vφ| = 5.89 deg·mth⁻¹, max|Vz| = 4.94 km·mth⁻¹
Sep:Oct 2010
altitude/km
latitude/deg
80
70
60
50
40
30
20
10
0
-90
-60
-30
0
30
60
90
5
4
3
2
1

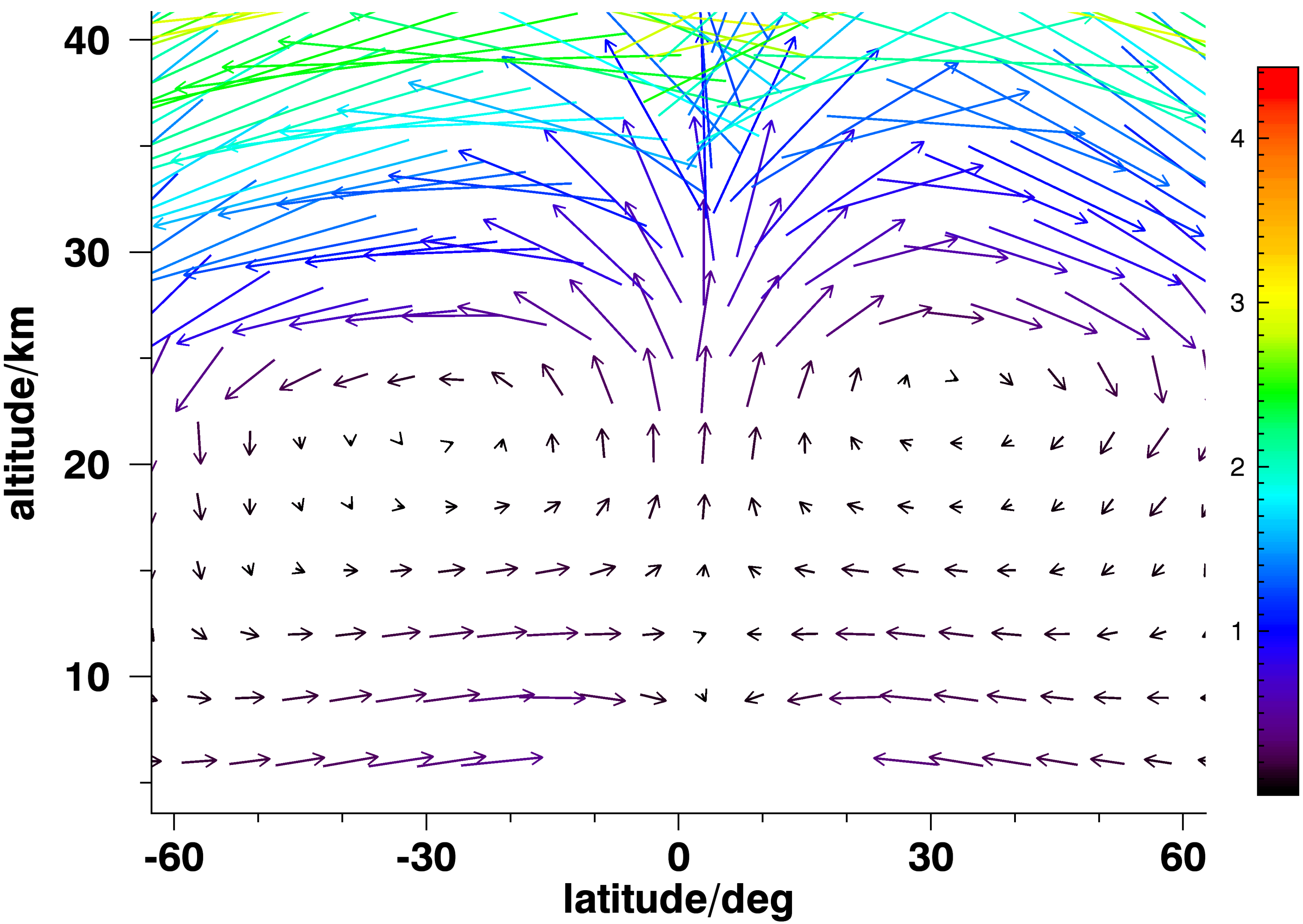


altitude/km
50
40
30
20
10
-60
-30
0
30
60
latitude/deg
5
4
3
2
1

# Test case 2:
# Annual mean vmr fields 2010
# From MIPAS
# Sinks/sources only– stationary atmosphere



Vφ,Vz, t=1, iter=40
max|Vφ| = 4.42 deg·mth⁻¹, max|Vz| = 1.48 km·mth⁻¹
Sep:Oct 2010
altitude/km
latitude/deg
0
10
20
30
40
50
60
70
80
-90
-60
-30
0
30
60
90
1
2
3
4

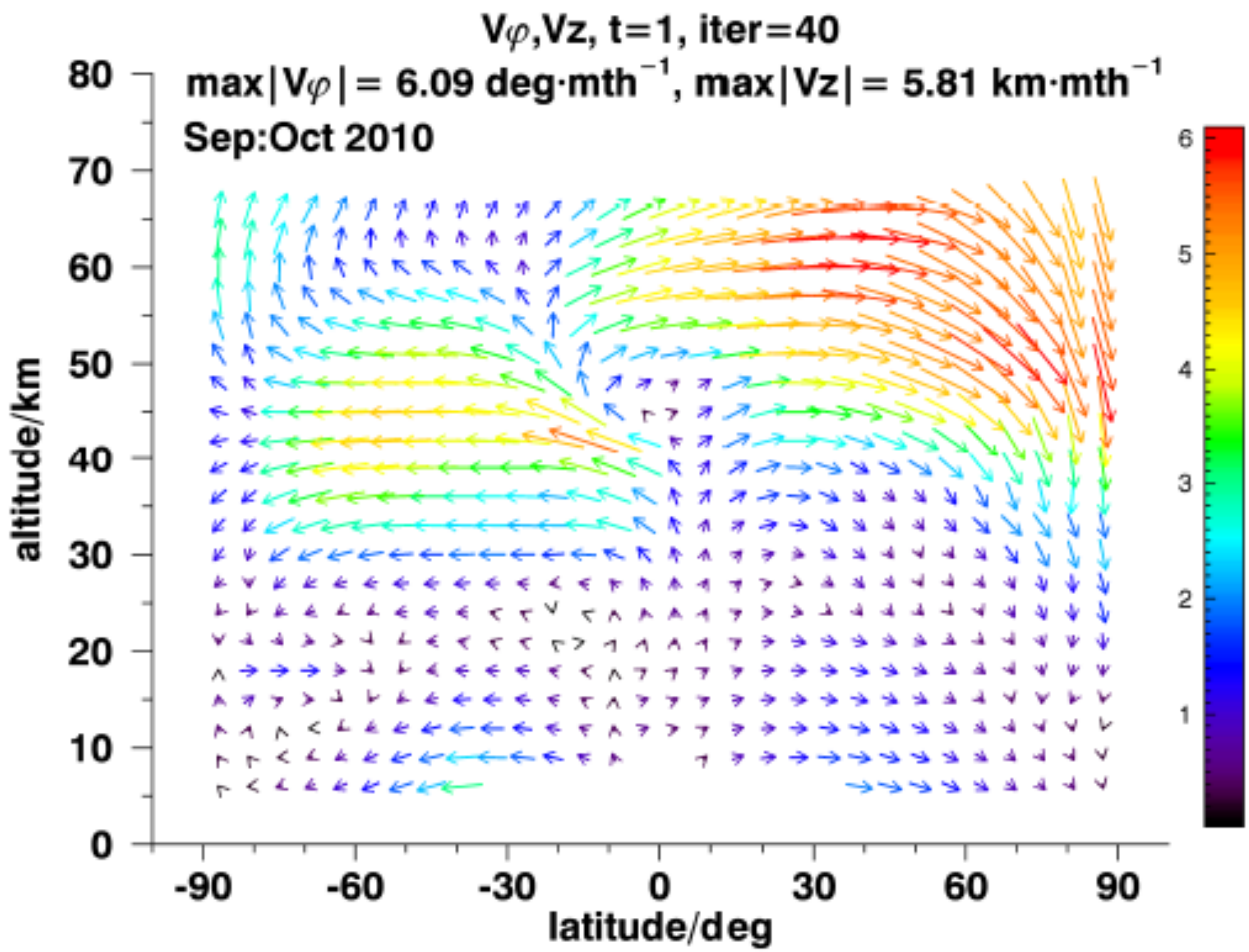


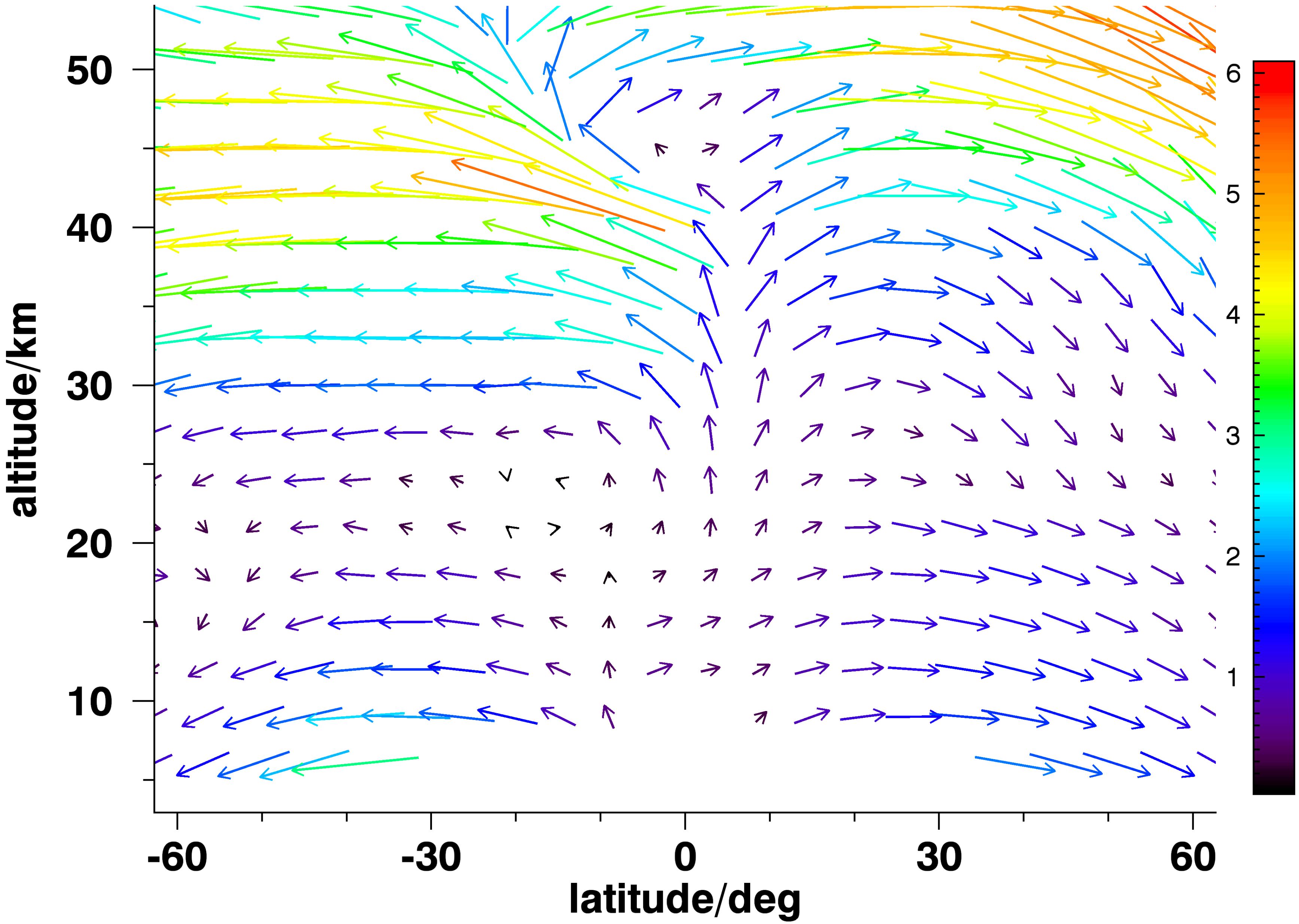
40
30
20
10
altitude/km
-60
-30
0
30
60
latitude/deg
4
3
2
1

# Test case 3:

# Sep-Oct 2010 vmr fields from MIPAS

# Inclusion of sinks and sources AND tracing vmr changes


Vφ,Vz, t=1, iter=40
max|Vφ| = 6.09 deg·mth⁻¹, max|Vz| = 5.81 km·mth⁻¹
Sep:Oct 2010
altitude/km
latitude/deg
80
70
60
50
40
30
20
10
0
-90
-60
-30
0
30
60
90
6
5
4
3
2
1



altitude/km
10
20
30
40
50
-60
-30
0
30
60
latitude/deg
1
2
3
4
5
6

# Next steps (as announced in Victoria-talk)

- Include more species;
- Consider sinks (photochemical, OH, $O^1D$);
- …

# Next steps

- Include more species; done!
- Consider sinks (photochemical, OH, $O^1D$); done!
- …

# Next steps

- Include more species; done!
- Consider sinks (photochemical, OH, $O^1D$); done!
- Apply for funding!

# THANK YOU

## ...and sorry for the maths

$$\frac{\ldots}{\partial v(\phi,z)} = -\left(\frac{\Delta t_p}{\Delta\phi}\right)^2\cdot\frac{1}{r^2}\cdot\left(v(\phi,z)+\frac{v(\phi-\Delta\phi,z)}{2}\right) \quad (46)$$

$$\frac{\partial\frac{\partial vmr_{i+1}(\phi+\Delta\phi,z)}{\partial vmr_i(\phi+\Delta\phi,z)}}{\partial v(\phi,z)} = -\left(\frac{\Delta t_p}{\Delta\phi}\right)^2\cdot\frac{v(\phi+\Delta\phi,z)}{2r^2} \quad (47)$$

$$\frac{\partial\frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi-\Delta\phi,z)}}{\partial v(\phi,z)} = \frac{1}{2r}\cdot\frac{\Delta t_p}{\Delta\phi}\cdot\left(1+\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi-\Delta\ldots)}{r}\right) \quad (48)$$

$$\frac{\partial\frac{\partial vmr_{i+1}(\phi+\Delta\phi,z)}{\partial vmr_i(\phi,z)}}{\partial v(\phi,z)} = \frac{1}{2r}\left(\frac{\Delta t_p}{\Delta\ldots}\right)^2\frac{v(\phi+\Delta\phi,z)}{r} \quad (49)$$

$$\frac{\partial\frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi+\Delta\phi,z)}}{\partial v(\phi,z)} = -\frac{\Delta t_p}{\Delta\phi}\cdot\frac{1}{r}\left(\frac{1}{2}-\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi,z)}{r}\right) \quad (50)$$

ntries not mentioned here are zero. Entries relevant to $w$ re:

$$\partial\frac{\partial\rho_{i+1}(\phi,z)}{\ldots} \quad \Delta t_p \quad \Delta t_p \quad v(\phi,z) \ldots$$

$$\frac{\ldots}{\partial w(\phi,z)} = \frac{1}{2}\cdot\left(\frac{\Delta t_p}{\Delta z}\right)^2\cdot\frac{(r+\Delta z)^2}{r^2}\cdot w(\phi,z+\Delta z) \quad (56)$$

$$\frac{\partial\frac{\partial\rho_{i+1}(\phi,z-\Delta z)}{\partial\rho_i(\phi,z)}}{\partial w(\phi,z)} = \frac{1}{2}\cdot\frac{\Delta t_p}{\Delta z}\cdot\frac{r^2}{(r-\Delta z)^2}\cdot\left(-1+\frac{\Delta t_p}{\Delta\phi}\cdot\frac{v(\phi,z-\Delta z)}{r-\Delta z}+\frac{\Delta t_p}{\Delta z}\cdot w(\phi,z-\Delta z)\right) \quad (57)$$

$$\frac{\partial\frac{\partial\rho_{i+1}(\ldots+\Delta\phi,\ldots-\Delta z)}{\partial\rho_i(\phi,z)}}{\partial w(\phi,z)} = -\ldots\frac{\Delta t_p}{\Delta\phi}\ldots\cdot\frac{r^2}{(r-\Delta z)^2}\cdot\frac{\cos(\ldots)}{\cos(\phi+\Delta\phi)}\cdot\frac{\ldots(\phi,z\ldots)}{r-\Delta z} \quad (58)$$

$$\frac{\partial\frac{\partial\rho_{i+1}(\ldots,z+\Delta z)}{\partial\rho_i(\phi+\Delta\phi,\ldots)}}{\partial w(\phi)} = -\frac{1}{2}\cdot\frac{\Delta t\ldots}{\Delta\phi}\cdot\frac{\Delta t_p}{\Delta z}\cdot\frac{r^2}{(r+\Delta z)^2}\cdot\frac{\cos(\phi+\Delta\phi)}{\cos(\phi)}\cdot\frac{v(\phi+\Delta\phi,z)}{r} \quad (59)$$

$$\frac{\partial\frac{\partial vmr_{i+1}(\phi,z)}{\partial vmr_i(\phi,z)}}{\partial w(\phi,z)} = \quad (60)$$

$$\left(\Delta t_p\right)^2 \ldots w(\phi,z-\Delta z) \ldots$$