FUNDAMENTALS OF MAGNETOHYDRODYNAMICS (MHD)

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"Basic SIDC seminar"
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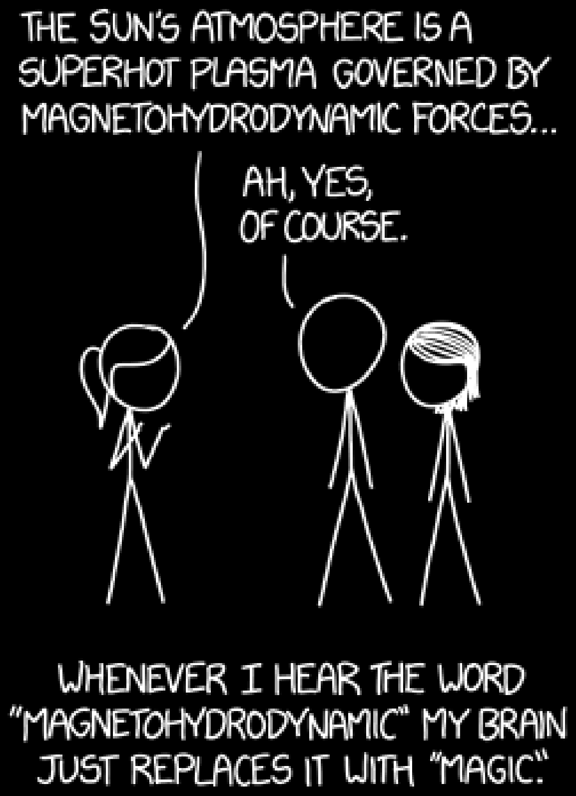

THE SUN'S ATMOSPHERE IS A SUPERHOT PLASMA GOVERNED BY MAGNETOHYDRODYNAMIC FORCES...
AH, YES, OF COURSE.
WHENEVER I HEAR THE WORD "MAGNETOHYDRODYNAMIC" MY BRAIN JUST REPLACES IT WITH "MAGIC."

1. IDEAL MHD

Sooo what is MHD?

certainly not magic

Hydrodynamics +

equations of gas dynamics + Maxwell's equations

1. IDEAL MHD

What about the “ideal” part?

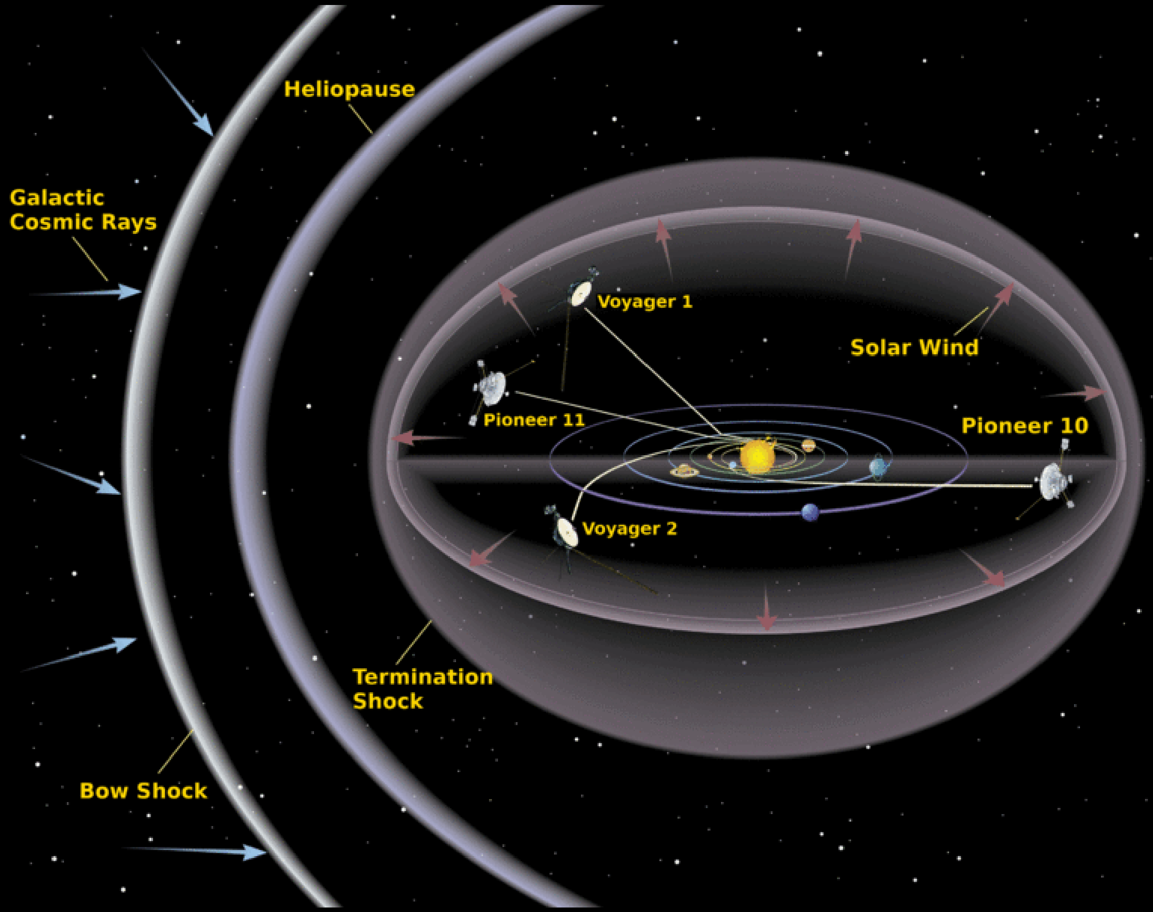
Assumptions:

- characteristic time >> ion gyroperiod and mean free path time
- characteristic scale >> ion gyroradius and mean free path length
- plasma velocities are not relativistic
- quasineutrality
- all dissipative processes (finite viscosity, electrical resistivity, thermal conductivity) are neglected

1. IDEAL MHD

Then… when is MHD useful?

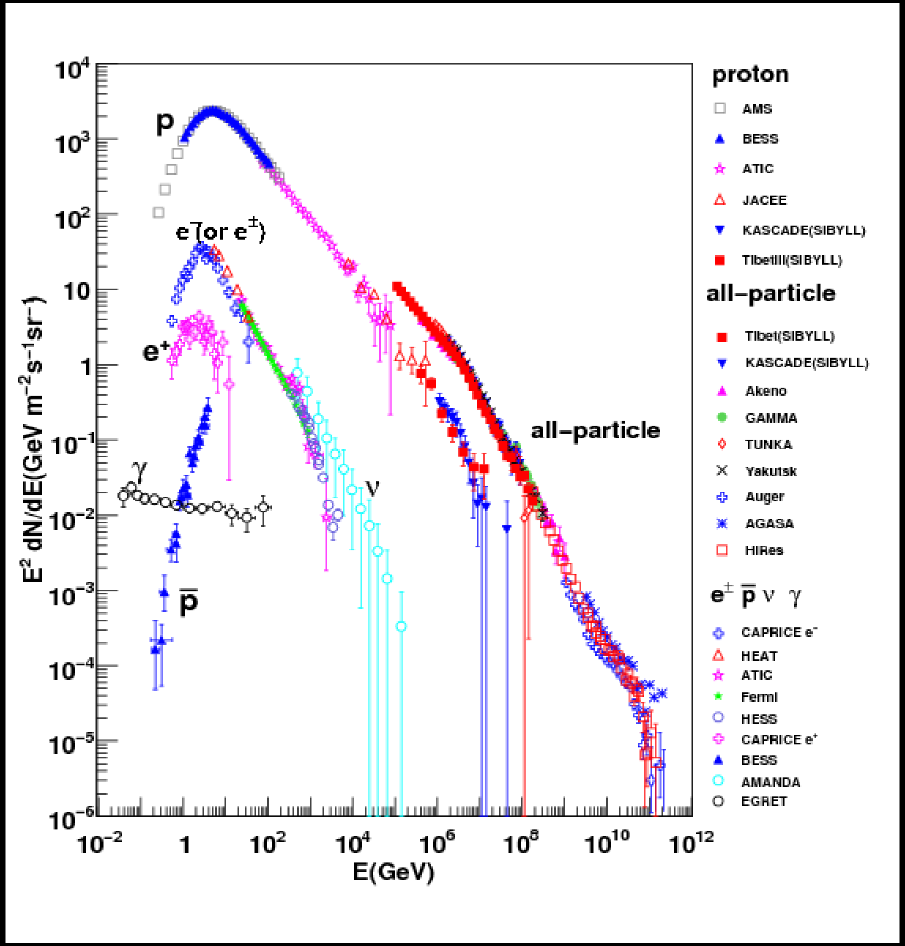
- describes macroscopic force balance, equilibria and dynamics on large scales
- MHD – good predictor of plasma stability
- systems described well by MHD:
  - solar wind, heliosphere, Earth's magnetosphere (large scales)
  - neutron star magnetospheres
  - inertial range of plasma turbulence



1. IDEAL MHD

When is MHD not useful?

- when non-fluid or kinetic effects are important
- the particle distribution functions are not Maxwellian (e.g. cosmic rays)
- the plasma is weakly ionized
- small scale plasmas



Cosmic ray spectra. Credit: Hongbo Hu, 2009

2. IDEAL MHD EQUATIONS

- describe the motions of a <u>perfectly</u> conducting fluid interacting with a magnetic field

- conservative form: $\frac{\partial}{\partial t}(\ldots) + \nabla \cdot (\ldots) = 0$; $\nabla \cdot \mathbf{B} = 0$ (no magnetic monopoles)

Mass conservation $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Momentum equation $$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \left[\rho \mathbf{v}\mathbf{v} + \left(p + \frac{1}{2}B^2\right)\mathbf{I} - \mathbf{B}\mathbf{B}\right] = \mathbf{0}, \qquad p = (\gamma - 1)\rho e$$

Energy conservation $$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e + \frac{1}{2}B^2\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho e + p + B^2\right)\mathbf{v} - \mathbf{v} \cdot \mathbf{B}\mathbf{B}\right] = 0$$

Magnetic flux conservation $$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}) = 0$$

2.1 MASS CONSERVATION

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

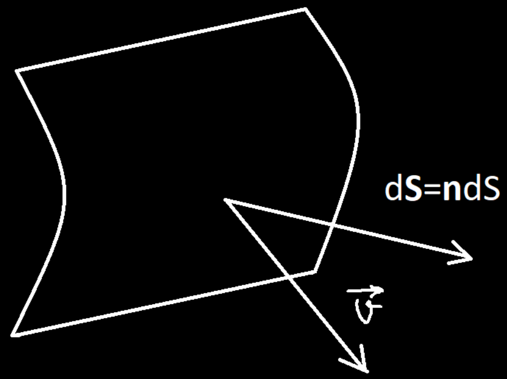
Net influx of mass

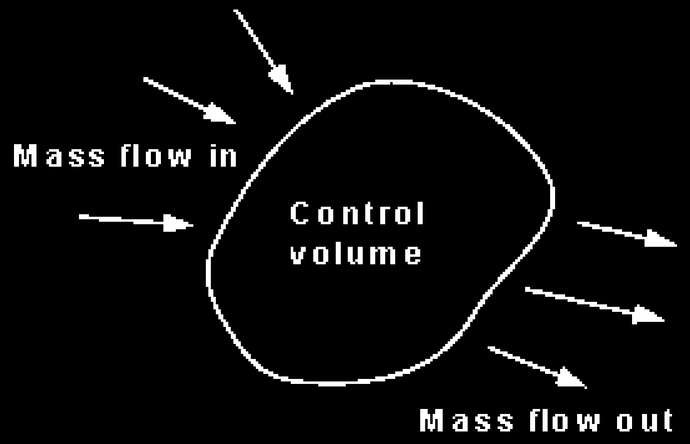
Rate of increase of mass (density) in control volume

ρ = mass density

v = flow velocity

ρ**v**S = mass flux passing through the surface





Input = Output + Accumulation,

in our case with Accumulation = 0

2.2 MOMENTUM EQUATION

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot\left[\overbrace{\rho\mathbf{vv} + \left(p+\frac{1}{2}B^2\right)\mathbf{I} - \mathbf{BB}}^{\text{Stress tensor}}\right] = \mathbf{0}$$

Time derivative of momentum density

Gradient of convective flux of momentum

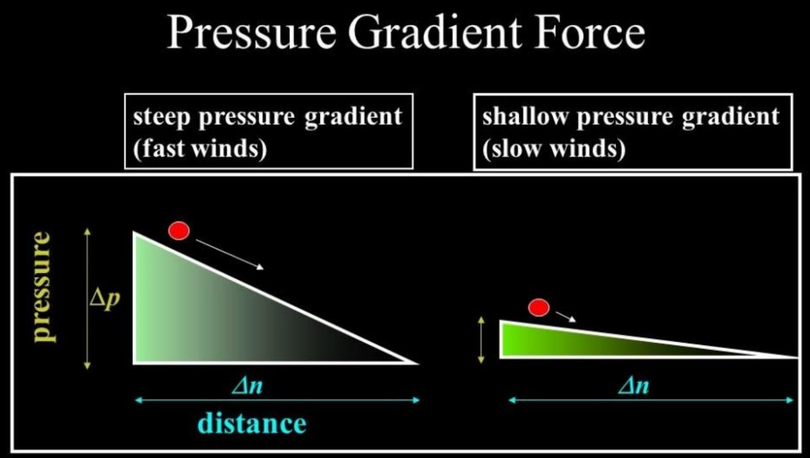
Total pressure (gradient)

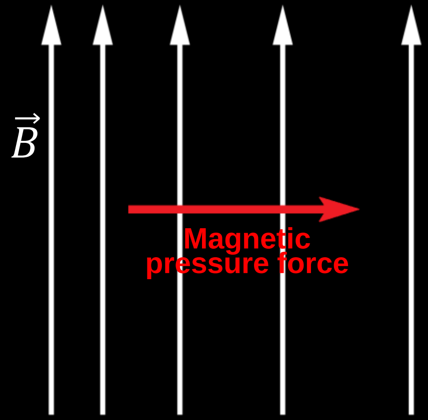
indicates isotropy

Magnetic tension

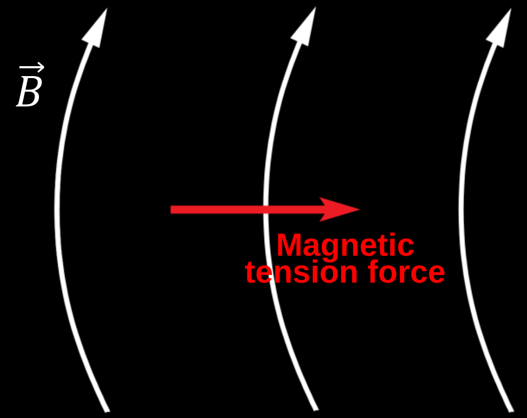
p = plasma pressure

$\frac{B^2}{2}$ = magnetic pressure





Regions of high magnetic pressure exert a force towards regions of low magnetic pressure



The magnetic tension force is directed radially inward with respect to magnetic field line curvature; it wants to straighten field lines

2.3 ENERGY CONSERVATION

$$\frac{\partial}{\partial t}\overbrace{\left(\frac{1}{2}\rho v^2 + \boxed{\rho e + \frac{1}{2}B^2}\right)}^{\text{Total energy density}} + \nabla\cdot\left[\overbrace{\left(\frac{1}{2}\rho v^2 + \rho e + p + B^2\right)\mathbf{v} - \mathbf{v}\cdot\mathbf{BB}}^{\text{Energy flow}}\right] = 0$$

Electromagnetic flow
(a.k.a. Poynting vector)

Internal energy density

Flow of kinetic energy

Kinetic energy density

Magnetic field energy density

Work done on the plasma from the pressure gradient

Flow of internal energy

$p = (\gamma - 1)\rho e$
γ = adiabatic index
e = internal energy per unit mass

Potential energy density = $\frac{p}{\gamma - 1} + \frac{1}{2}B^2$

(expressed in primitive variables)

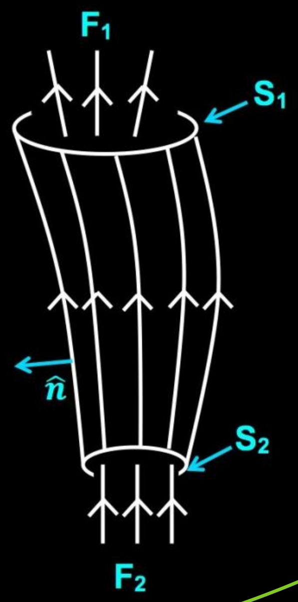
The entropy of any plasma element is constant.

2.4 MAGNETIC FLUX CONSERVATION, FROZEN-IN CONDITION

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \underbrace{(\mathbf{vB} - \mathbf{Bv})}_{\text{represents conversion of mechanical energy to electromagnetic induction}} = 0$$

Magnetic field time variation

some math in between…



Magnetic flux tube = cylindrical volume enclosed by the collection of field lines that intersect a closed curve

$$\frac{d}{dt}\left(\int_S \mathbf{B} \cdot d\mathbf{S}\right) = 0$$ Frozen-in condition

1) *The magnetic flux through a surface moving with the plasma is conserved.*
2) *Magnetic field lines behave as if they move with the plasma.*
3) *Magnetic topology is conserved.*

3. PLASMA β

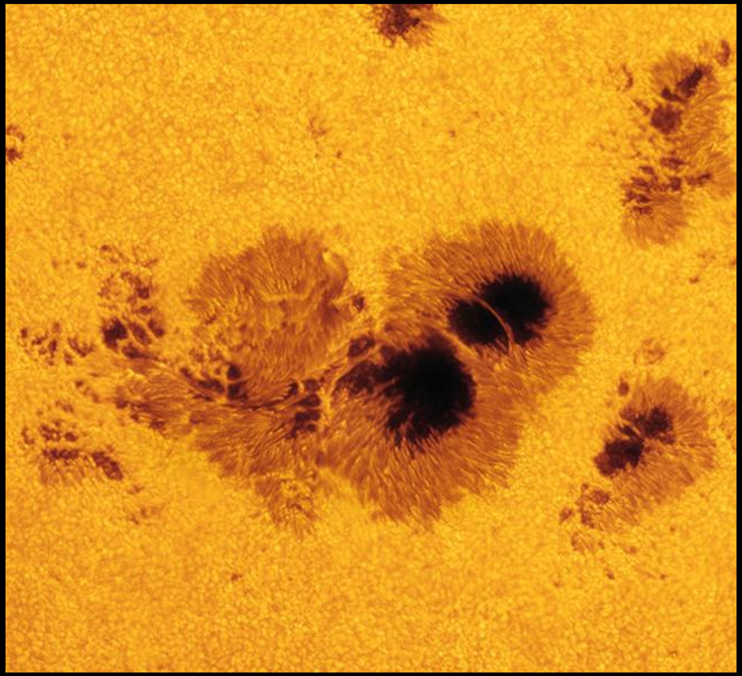
$$\beta = \frac{thermal\ pressure}{magnetic\ pressure} = \frac{2\mu_0 p_{th}}{B^2}$$

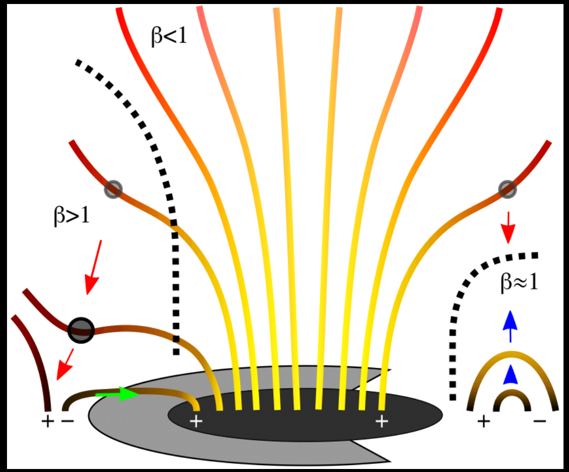
$$p_{th} = 2nKT;\ \ p_{mag} = \frac{B^2}{2\mu_0}$$

Both obey the frozen-in condition, the difference is which component is dominant

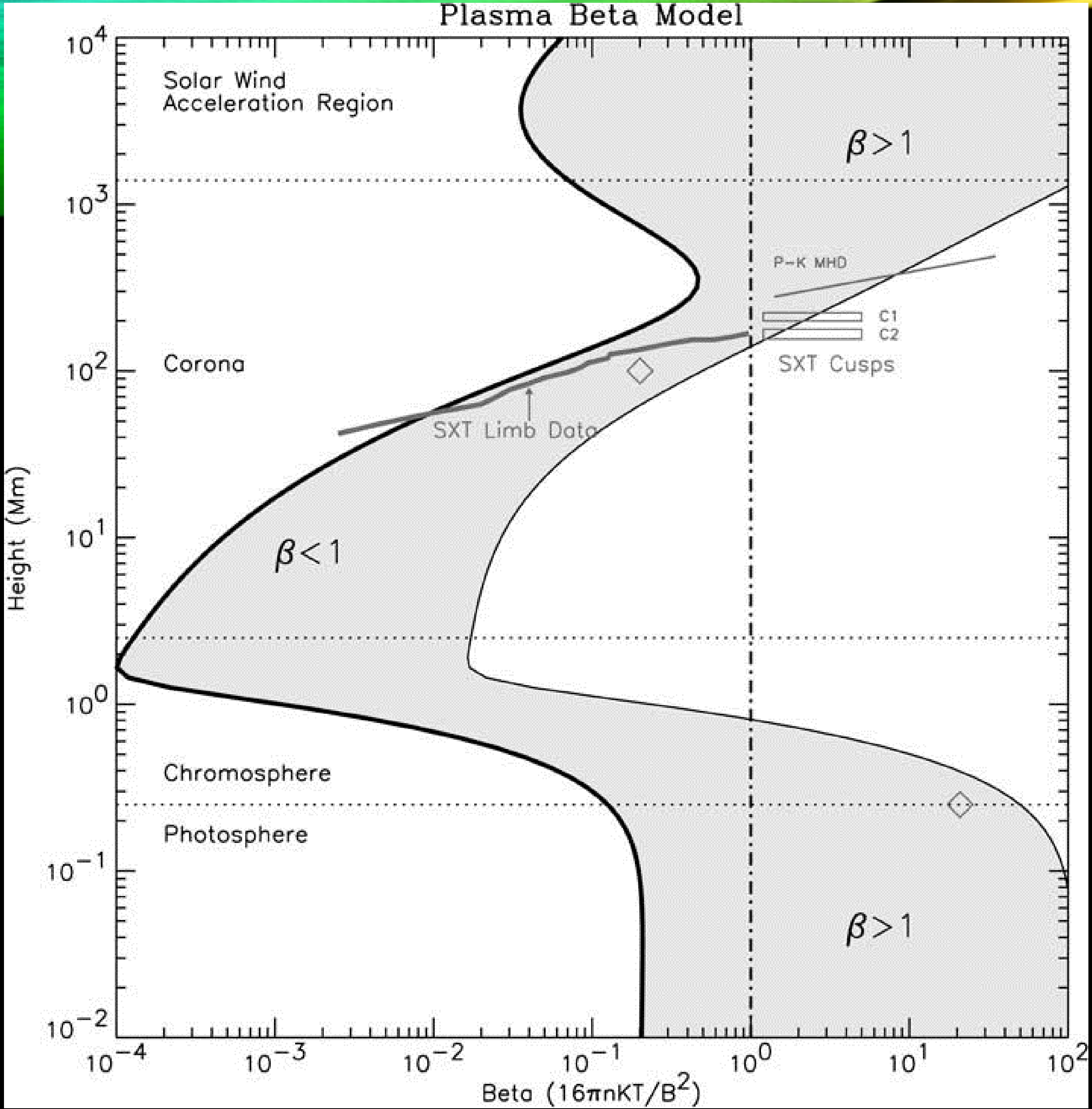
$\beta \ll 1$: magnetic field carries the plasma

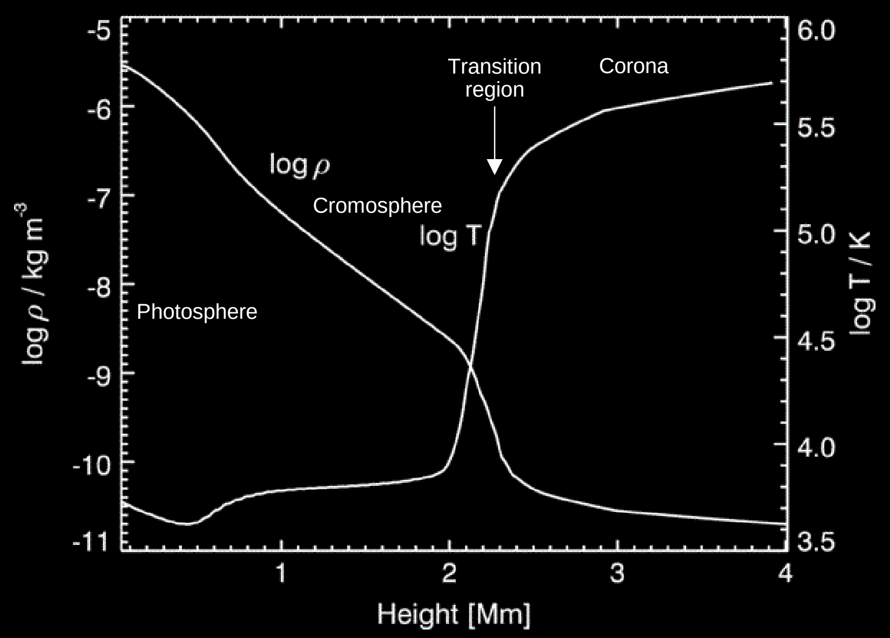
$\beta \gg 1$: plasma carries $\mathbf{B}$ as it moves





Sketch of a sunspot with forming penumbra.
Credits: Bourdin, 2017





Computed dependencies of logarithms of the density ρ and the temperature T. Credit: Fedun, Shelyag and Erdélyi, 2010

Plasma beta model over an active region. The plasma beta as a function of height is shown shaded for open and closed field lines originating between a sunspot of 2500 G and a plage region of 150 G.

Gary, 2001

4. ALFVÉN MACH NUMBER

$$\boxed{M_A = \frac{v}{v_A} = \sqrt{\frac{p_{ram}}{p_{mag}}}}$$

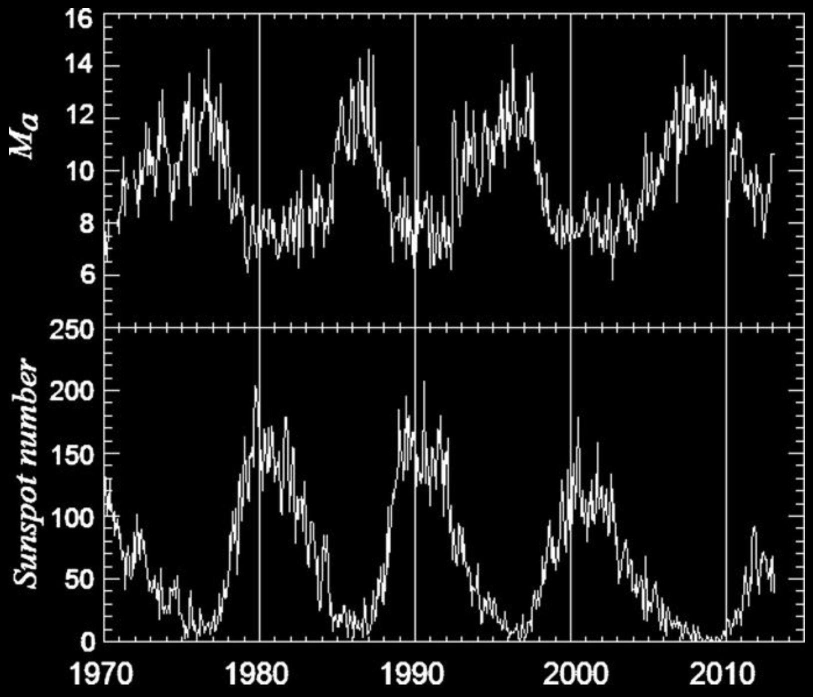
$$v_A = \frac{B}{\sqrt{\mu_0 \rho}} = Alfvén\ speed; \quad p_{ram} = \frac{\rho v_{sw}^2}{2}; \ p_{mag} = \frac{B^2}{2\mu_0}$$

basically, $M_A^2 = \dfrac{kinetic\ energy}{magnetic\ energy}$

$M_A < 1$: sub – Alfvénic flow/speeds

Low corona: low speeds, still strong magnetic field => sub-Alfvénic flows

$M_A > 1$: super – Alfvénic flow/speeds



Plot of the Alfvén Mach number (top) and sunspot number (bottom) as a function of time through the solar cycle using 27 day averaged OMNI2 data. Zank et al., 2014

Oooor this…

that feels like…

De Havilland Mosquito

5. SINGLE PARTICLE MOTION IN EM FIELDS

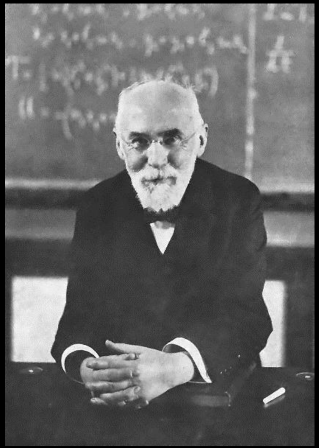
Lorentz force

Not this guy.
Definitely not.

5. SINGLE PARTICLE MOTION IN EM FIELDS

$$\boxed{m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

Lorentz force



This guy.

Simple case: **E** = 0

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}$$

then

Let's assume: $\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$ $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

then

We calculate first and second derivatives of the speed and solve the differential equation system and voila!

$$v_x = A_1 cos\left(\frac{qB}{m}t\right) + B_1 sin\left(\frac{qB}{m}t\right)$$

$$v_y = A_2 cos\left(\frac{qB}{m}t\right) + B_2 sin\left(\frac{qB}{m}t\right)$$

also

$$v_x(0) = v_0$$
$$v_y(0) = 0$$
$$\dot{v_x}(0) = 0$$

and

We define: $\omega_c = \frac{qB}{m}$ cyclotron frequency

5. SINGLE PARTICLE MOTION IN EM FIELDS

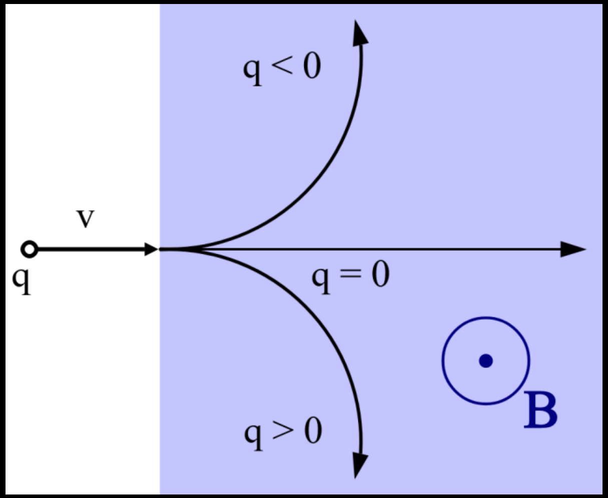
$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Some mathematical magic…

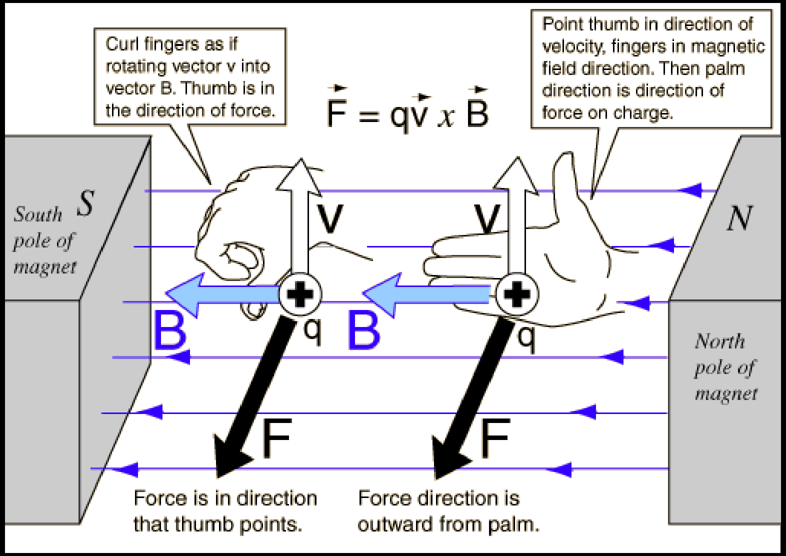
Solution: $v_x = v_0 \cos(\omega_c t)$, $v_y = -v_0 \sin(\omega_c t)$ and $X = \boxed{\frac{v_0}{\omega_c}} \sin(\omega_c t)$, $Y = \frac{v_0}{\omega_c} cos(\omega_c t)$

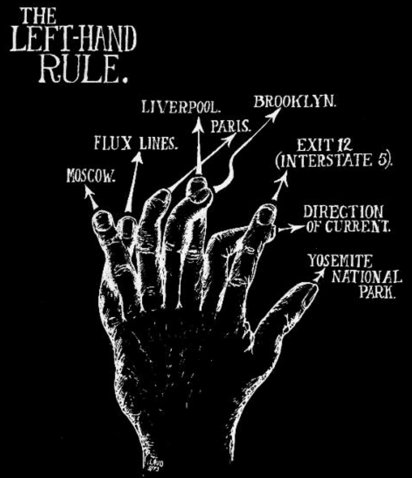
gyroradius or cyclotron radius

CIRCULAR MOTION!

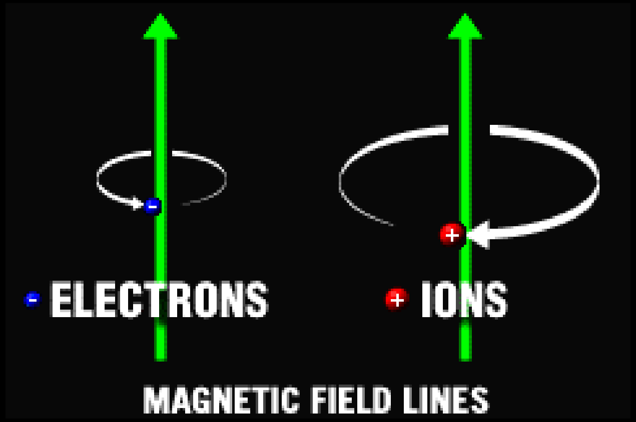


Right hand rule





Make sure you don't end up like this…



5. SINGLE PARTICLE MOTION IN EM FIELDS

$$\boxed{m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

Lorentz force

Let's consider

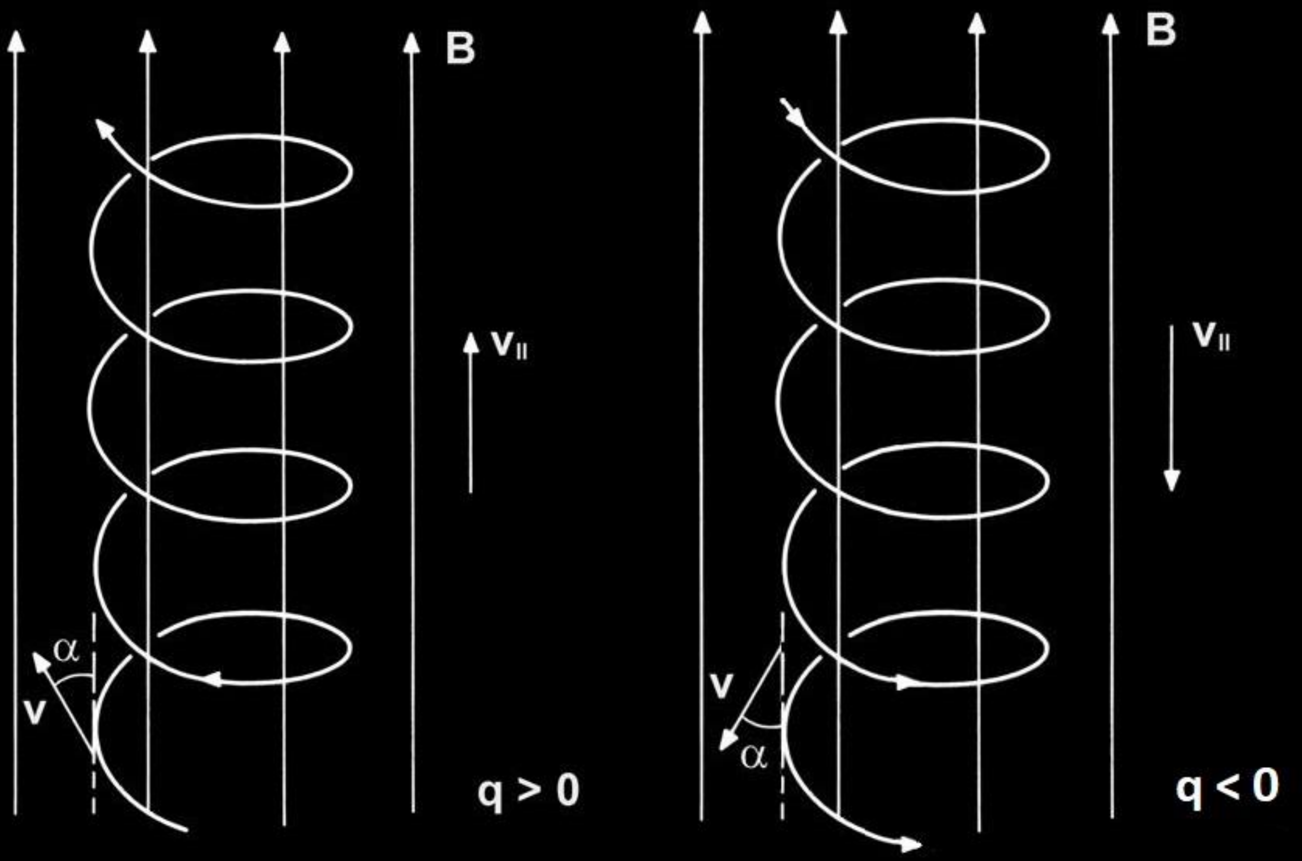
$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \quad \mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

but the components of the initial velocity parallel and perpendicular to **B** are non zero.

α = pitch angle

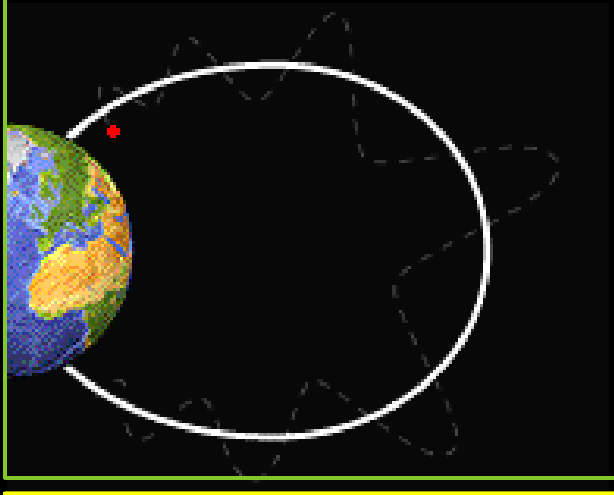
Motion?

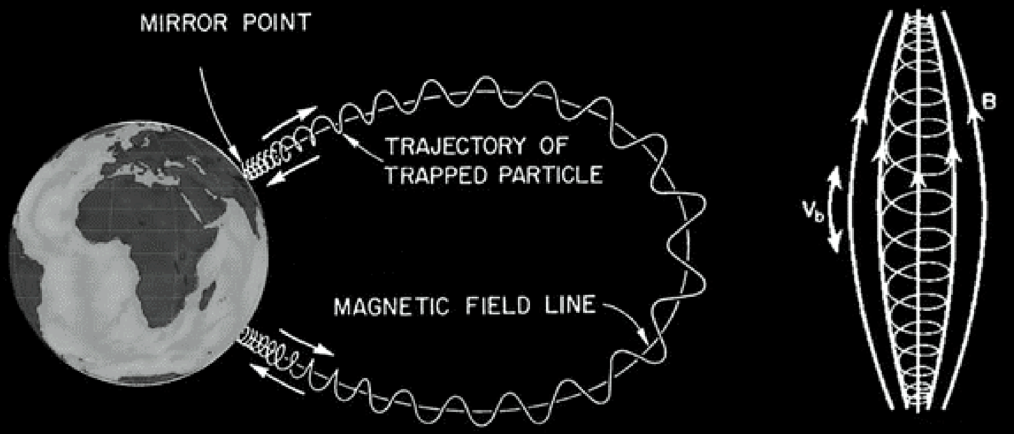
<u>along</u> the magnetic field lines!



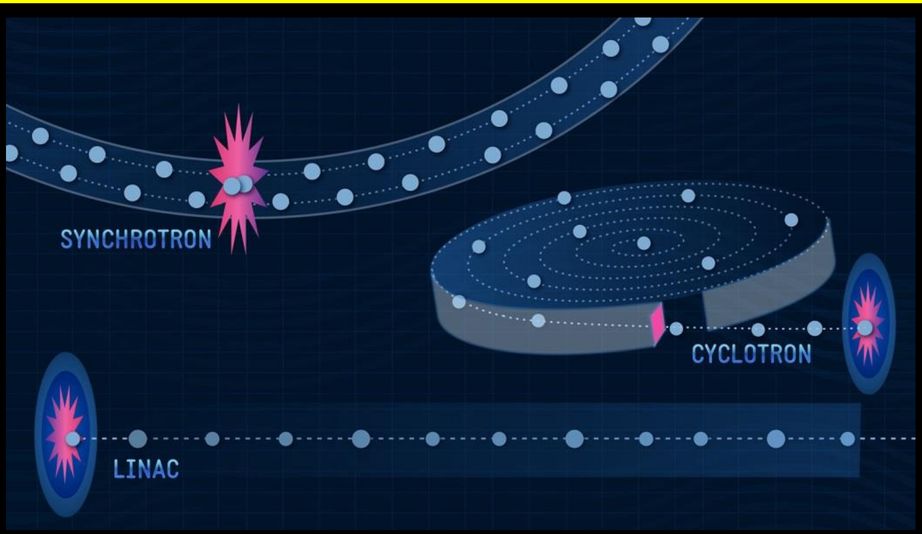
5. SINGLE PARTICLE MOTION IN EM FIELDS

Applications? **Countless!**

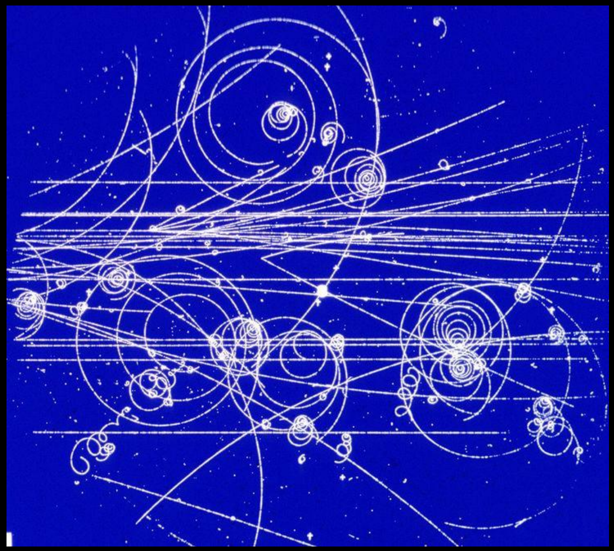




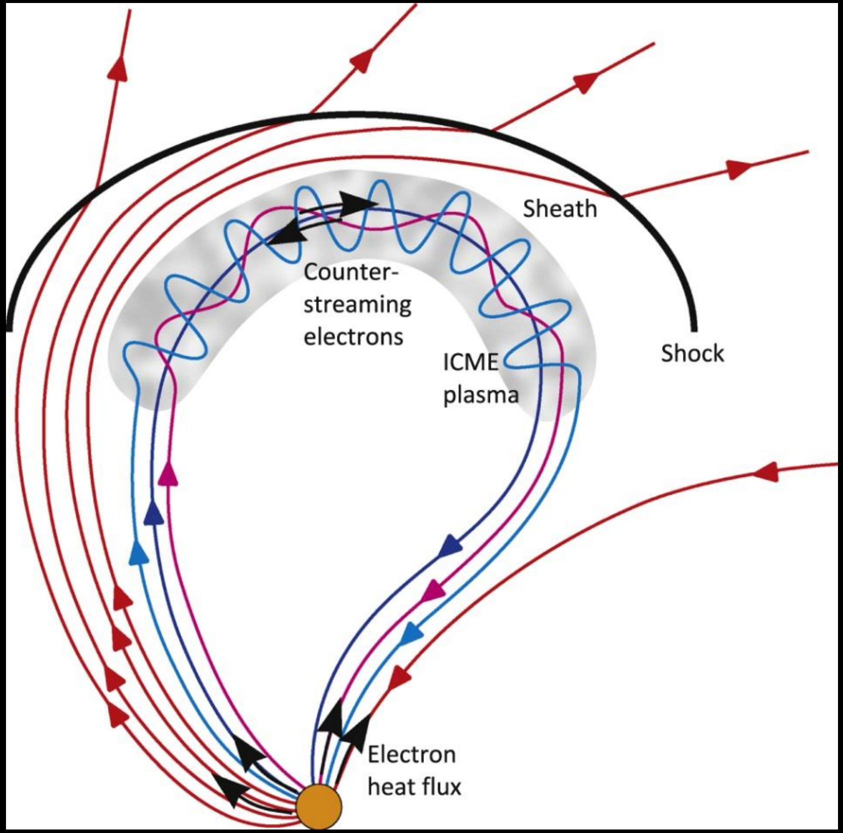
Particles trapped inside Earth's magnetic field lines; magnetic mirroring



Accelerators



Bubble and cloud chambers



Electrons inside magnetic flux ropes of ICMEs. Owens 2016

5. SINGLE PARTICLE MOTION IN EM FIELDS

$$\boxed{m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$ Lorentz force

Let's add also **E** perpendicular to **B**; we make the substitution: $\boxed{\mathbf{v} = \boldsymbol{\omega} + \mathbf{E} \times \mathbf{B}}$

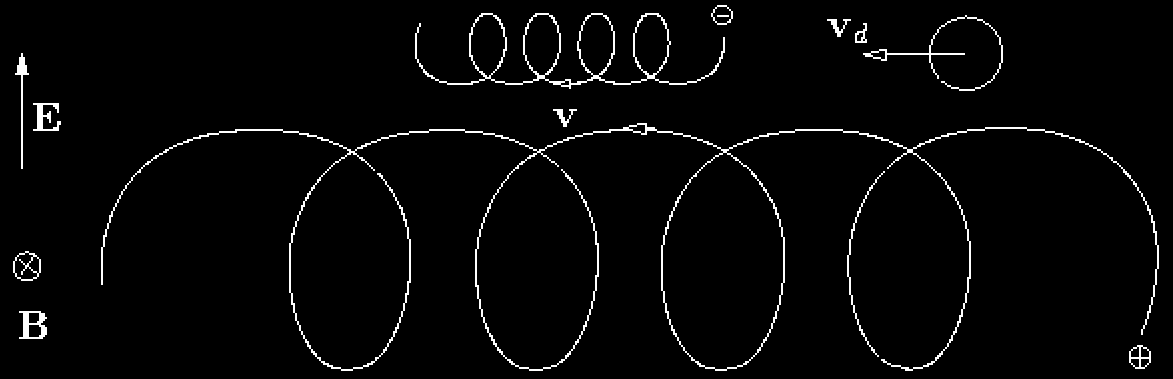
$\Rightarrow$ $\mathbf{E} \times \mathbf{B}$ drift, with a speed:

$$\boxed{\mathbf{v_D} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}}$$

$\Rightarrow$ $\mathbf{E} \times \mathbf{B}$ drift independent of particle mass, charge or speed!

Motion?

drift **<u>across</u>** the magnetic field lines



5. SINGLE PARTICLE MOTION IN EM FIELDS

$$\boxed{m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

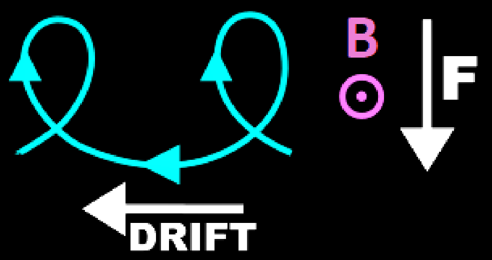
Lorentz force

What about gravity?

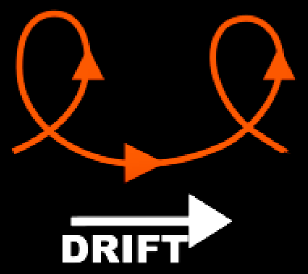
$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} + q(\mathbf{v} \times \mathbf{B}) = q(\frac{1}{q}\mathbf{F} + \mathbf{v} \times \mathbf{B}) \quad \Rightarrow \text{drift speed:} \quad \boxed{\mathbf{v_D} = \frac{1}{q}\frac{\mathbf{F} \times \mathbf{B}}{B^2}}$$

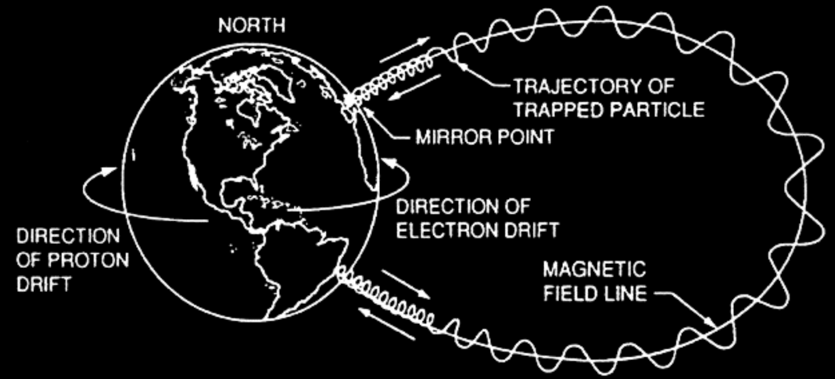
Motion? ⇒ drift across the magnetic field lines, dependent on charge!

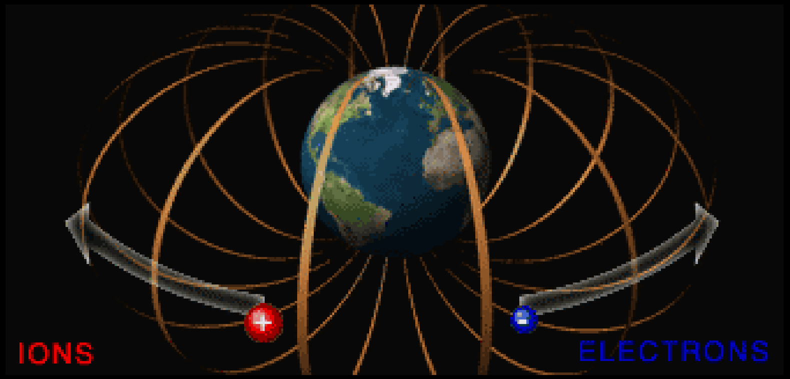
positives +



negatives -

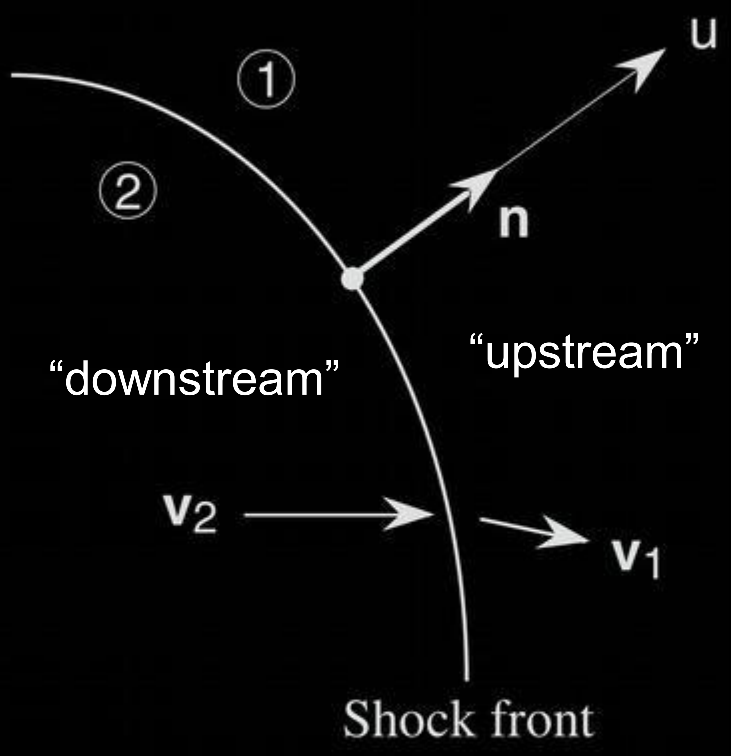






Curvature drift
important in this case!

6. SHOCKS AND DISCONTINUITIES



= surface separating two fluids (or gases) with different physical properties, in equilibrium

Notation: $[\![f]\!] = f_1 - f_2$
indices "n" and "t" will denote the components of a vector normal and tangential to the surface, and indices 1 and 2 the different media upstream or downstream of the shock front

Variables: ρ, v_n, $\mathbf{v_t}$, p, B_n, $\mathbf{B_t}$; the discontinuity/shock type is determined by the variables that jump (vary) across the surface

Difference?

$[\![v_n]\!] = 0$

Types of discontinuities:

1a. Contact
1b. Tangential } no mass flow
1c. Rotational → mass flows

$[\![v_n]\!] \neq 0$

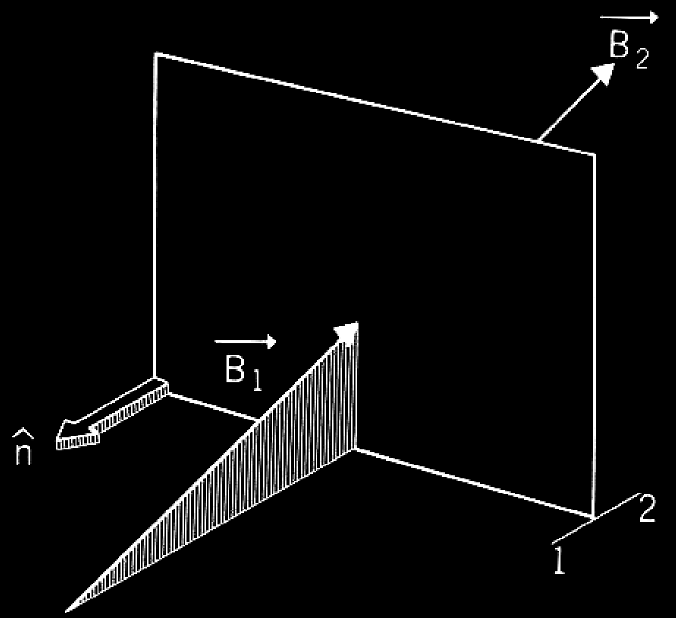
Types of shocks:

2a. Slow
2b. Fast
2c. Intermediate } mass flows through boundary

6.1 DISCONTINUITIES

1a. Contact discontinuity

= boundary between two media which have different densities and temperatures; no flow of mass across it



$B_n \neq 0$, field lines can cross the discontinuity

Jumping: $[\![\rho]\!] \neq 0$

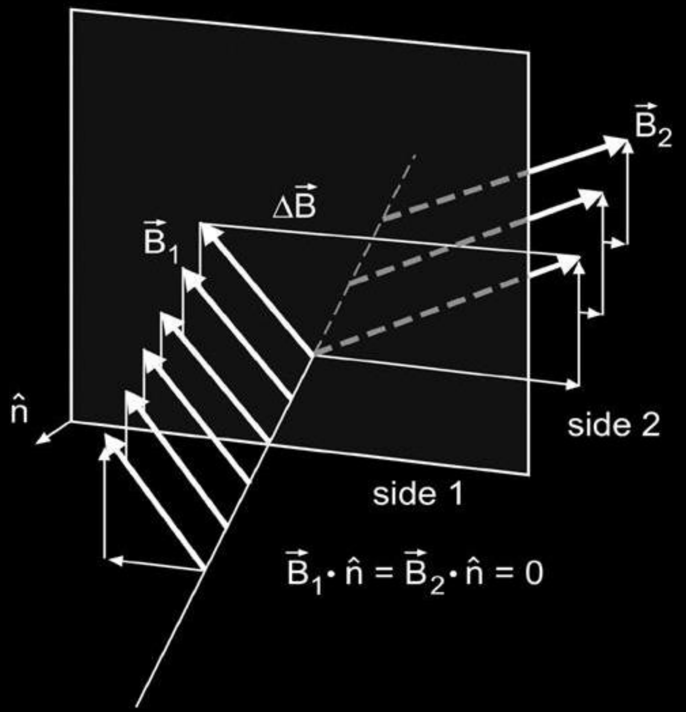
Continuous: $v_n = 0, [\![\mathbf{v_t}]\!] = 0, [\![p]\!] = 0, [\![B_n]\!] = 0, [\![\mathbf{B_t}]\!] = 0$

no mass flow across the surface

6.1 DISCONTINUITIES

1b. Tangential discontinuity

- no mass flow, no magnetic flux across it

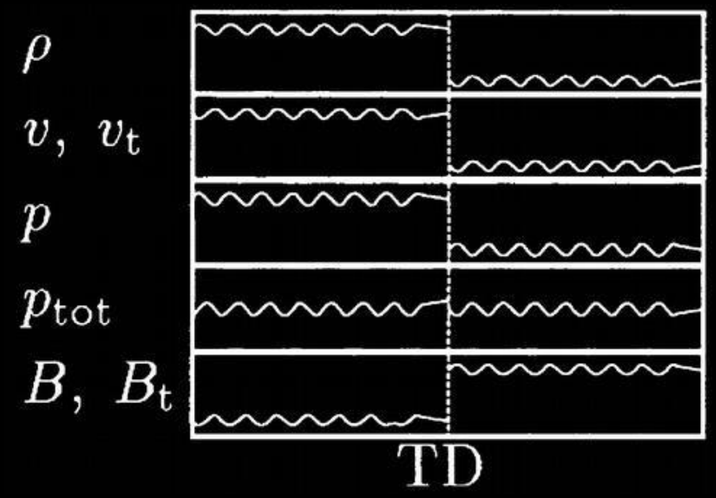


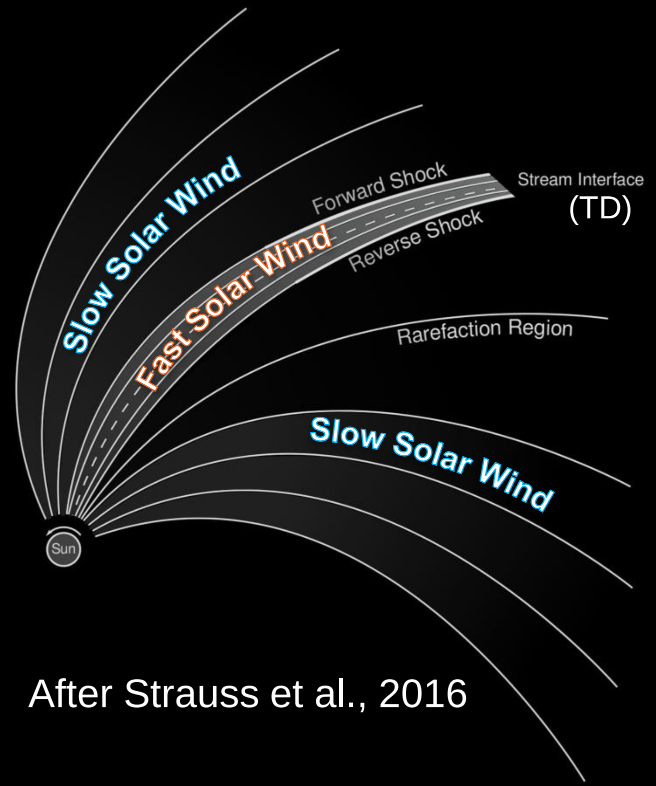
After Burlaga and Ness, 1969

$B_n = 0$, field lines do not cross the discontinuity; upstream and downstream magnetic field vectors are parallel to the shock plane

Jumping: $[\![\rho]\!] \neq 0, [\![\mathbf{v_t}]\!] \neq 0, [\![p]\!] \neq 0, [\![\mathbf{B_t}]\!] \neq 0$

Continuous: $v_n = 0, B_n = 0, \left[\!\!\left[\boldsymbol{p} + \frac{\boldsymbol{B_t^2}}{\boldsymbol{2\mu_0}}\right]\!\!\right] = 0$





After Strauss et al., 2016

6.1 DISCONTINUITIES

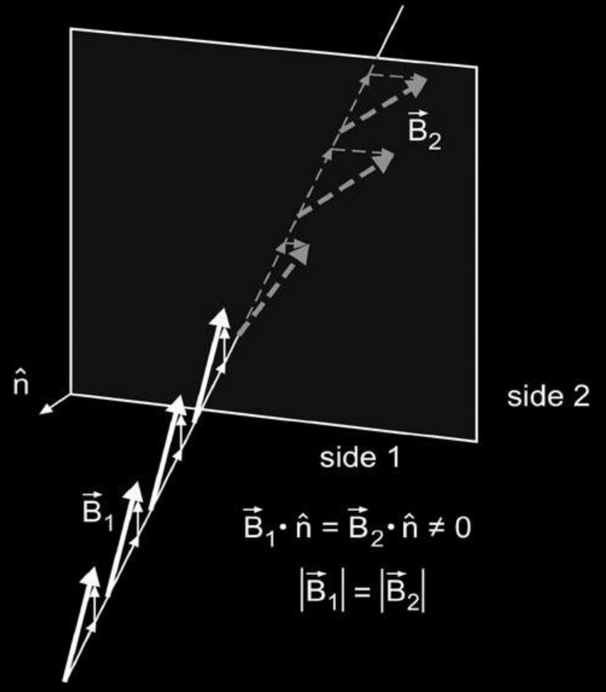
1c. Rotational discontinuity

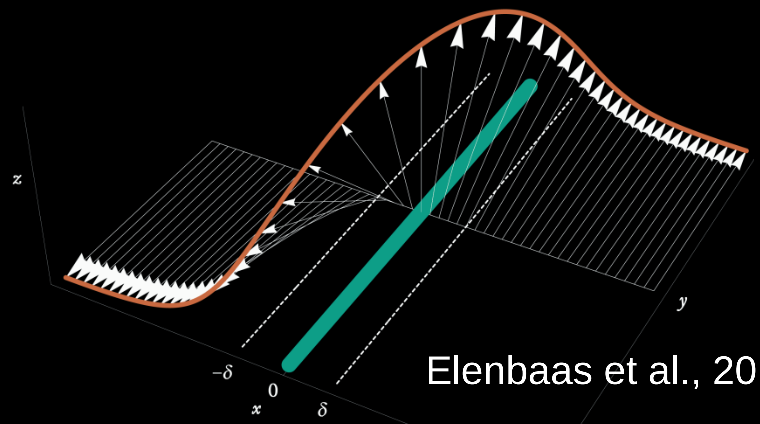
= actually a particular case of intermediate shock, in which $[\![\rho]\!] = 0$; plasma flows across the surface

All thermodynamic quantities are continuous across the shock, but the tangential components $\mathbf{B_t}$ and $\mathbf{v_t}$ can rotate (but same magnitude).

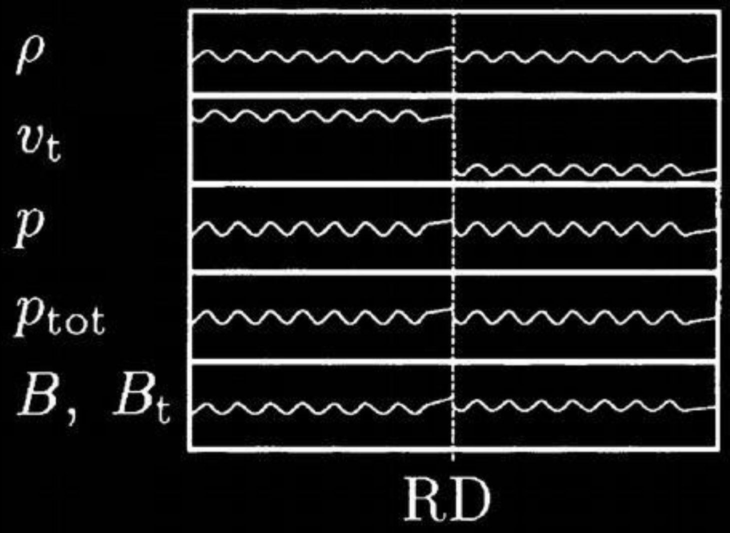
Jumping: $[\![\mathbf{v_t}]\!] \neq 0, [\![\mathbf{B_t}]\!] \neq 0$
Continuous: $[\![\rho]\!] = 0, [\![v_n]\!] = 0, [\![p]\!] = 0, [\![B_n]\!] = 0$

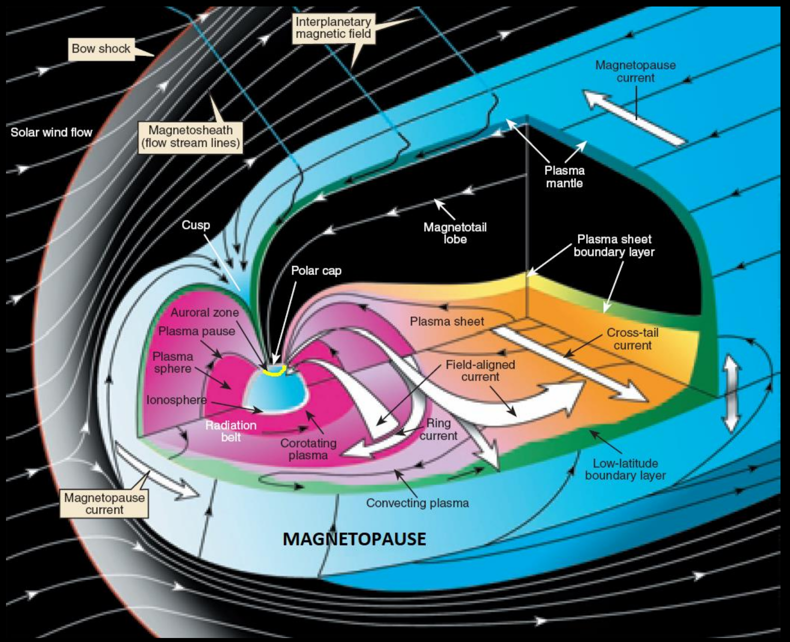
Propagation speed = normal Alfvén speed = $\dfrac{B_n}{\sqrt{\rho\mu_0}}$



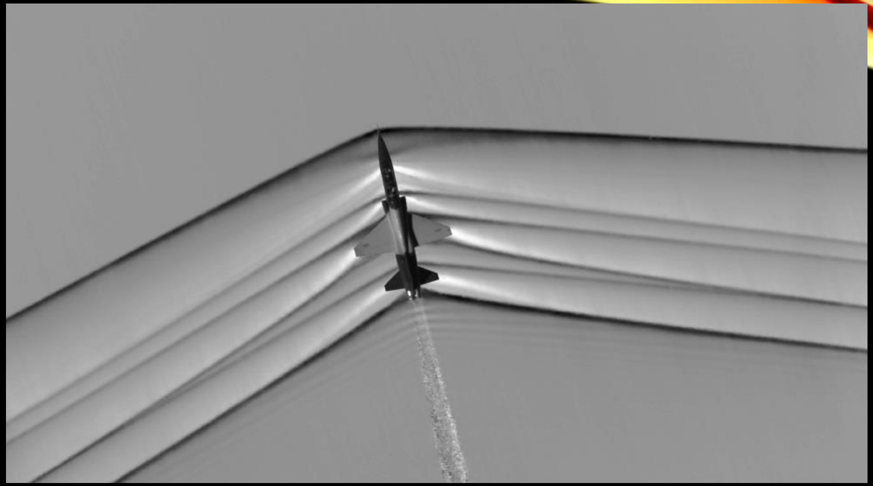


Elenbaas et al., 2015





6.2 SHOCKS



= discontinuities across which there is a flux of mass; they are created when the plasma is moving with a speed higher than the information can propagate and encounters an obstacle.
All MHD shocks have the property of coplanarity = downstream magnetic field lies in the plane defined by the upstream magnetic field and the shock normal

Jumping: $[\![\rho]\!] \neq 0, [\![v_n]\!] \neq 0, [\![p]\!] \neq 0, [\![\mathbf{v_t}]\!] \neq 0, [\![\mathbf{B_t}]\!] \neq 0$
Continuous: $[\![B_n]\!] = 0$

Remember Alfvén speed? $v_A = \frac{B}{\sqrt{\mu_0 \rho}}$

Another one! Sound speed: $C_s = \sqrt{\gamma \frac{p}{\rho}}$

Since plasma is a fluid interacting with the magnetic field → characteristic speeds = combinations of Alfvén and sound speeds

6.2 SHOCKS

Three types of waves propagate in the magnetized solar wind plasma. They are ordered by their characteristic speeds (phase velocities), which are called fast, intermediate, and slow (v_{fast}, v_i, v_{slow}, respectively) and they are defined as follows:

$$v_{fast/slow}^2 = \frac{1}{2}\left[(C_s^2 + v_A^2) \pm \sqrt{(C_s^2 + v_A^2)^2 - 4C_s^2 v_A^2 cos^2\theta_{Bn}}\right], \qquad v_i = v_A cos\theta_{Bn}$$

$$\theta_{Bn} = \text{angle between the incoming magnetic field and the shock normal vector}$$

For any θ, C_s and v_A, $v_{fast} \geq v_A \geq v_{slow}$.

3 types of shocks, depending on the speed of the incoming flow:

$\longrightarrow$ $v > v_{fast}$ $\Rightarrow$ **fast** shock

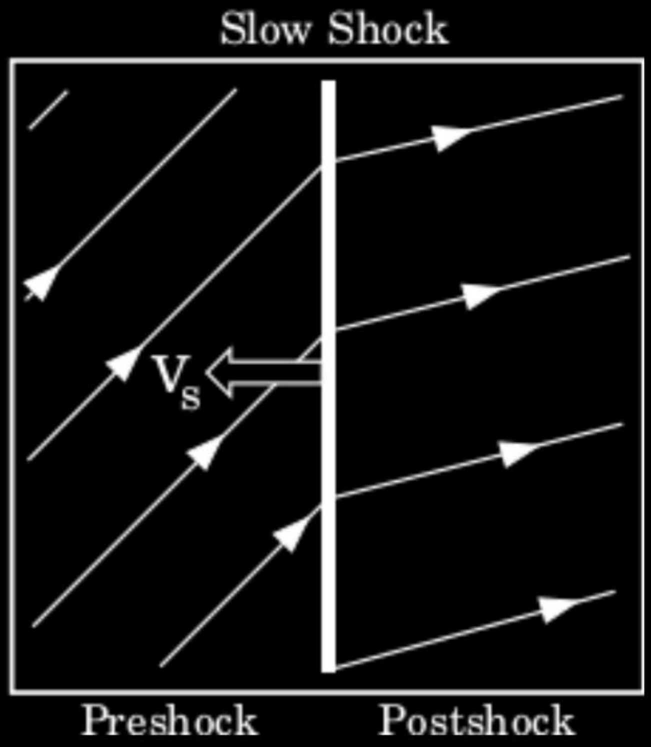
$\longrightarrow$ $v > v_{slow}$ $\Rightarrow$ **slow** shock

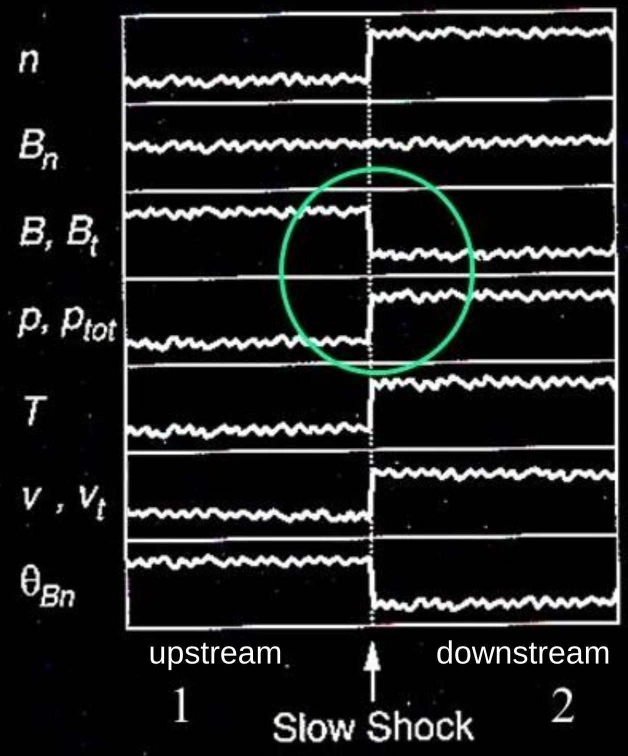
$\left.\begin{array}{l}\longrightarrow v > v_A \\ \quad v < v_{fast}\end{array}\right\}$ $\Rightarrow$ **intermediate** shock

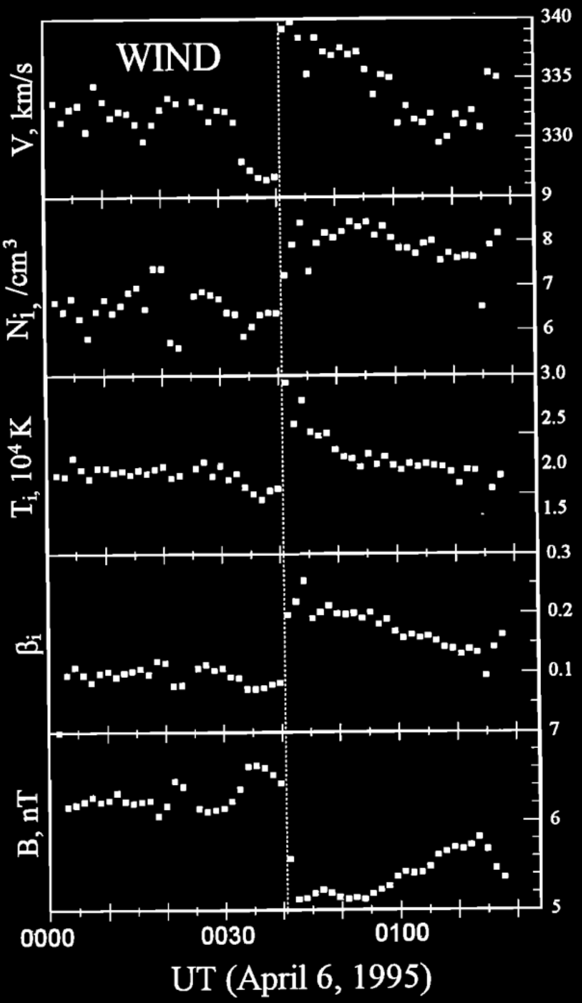
2a. Slow (magnetoacoustic) shocks

$v > v_{slow}$

→ Magnetic field decreases and gets refracted towards the shock normal
→ Plasma pressure increases







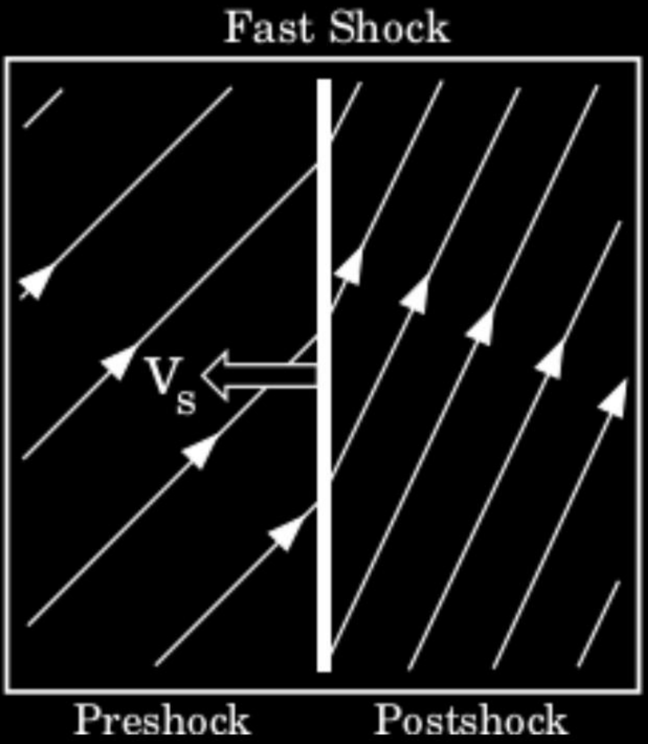
Slow shock observed in Wind magnetic field and proton data on April 6, 1995. Credits: Wang et al., 1998

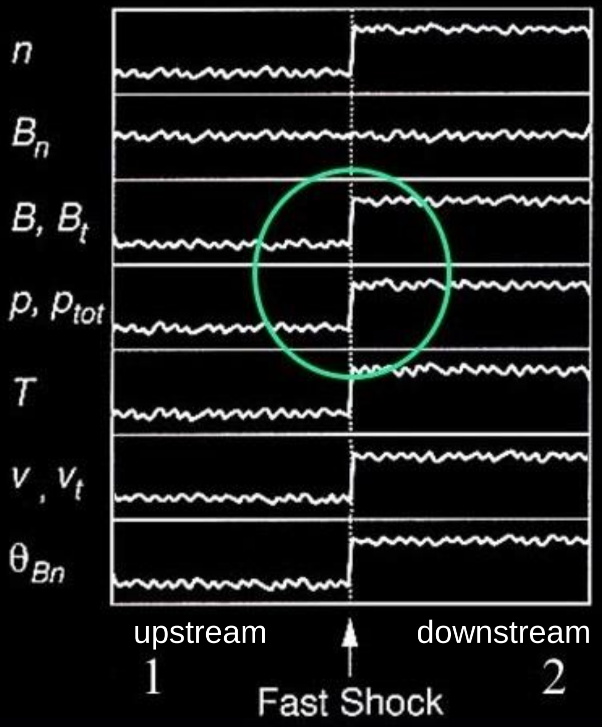
6.2 SHOCKS

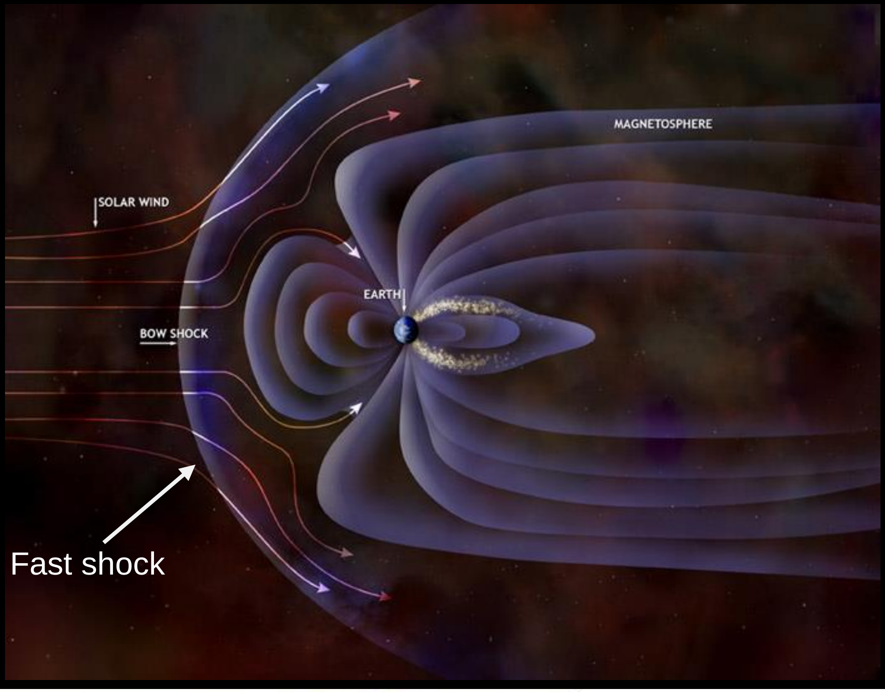
2b. Fast (magnetoacoustic) shocks

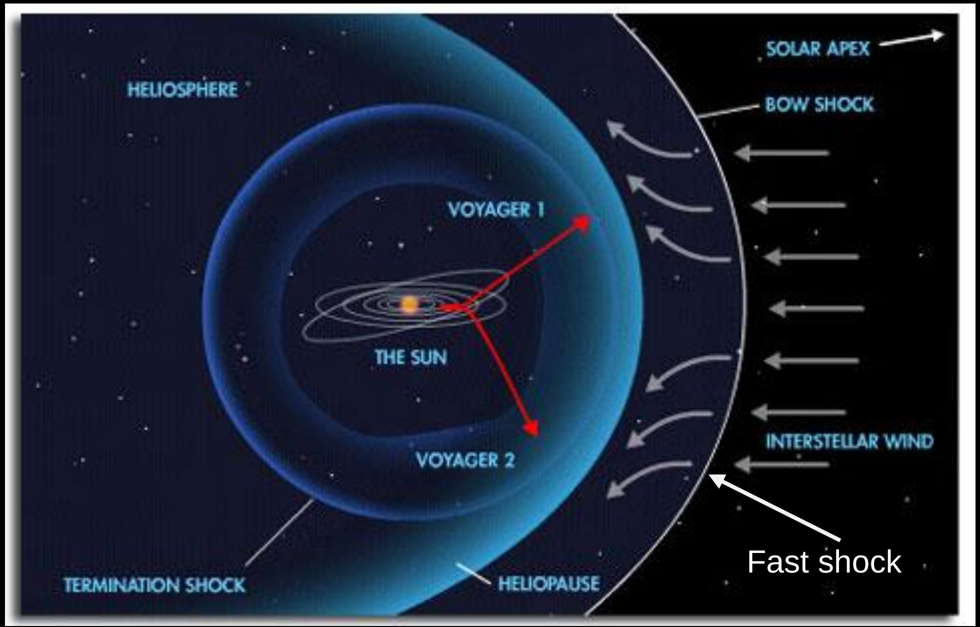
$v > v_{fast}$

→ Magnetic field increases and gets refracted away from the shock normal
→ Plasma pressure increases









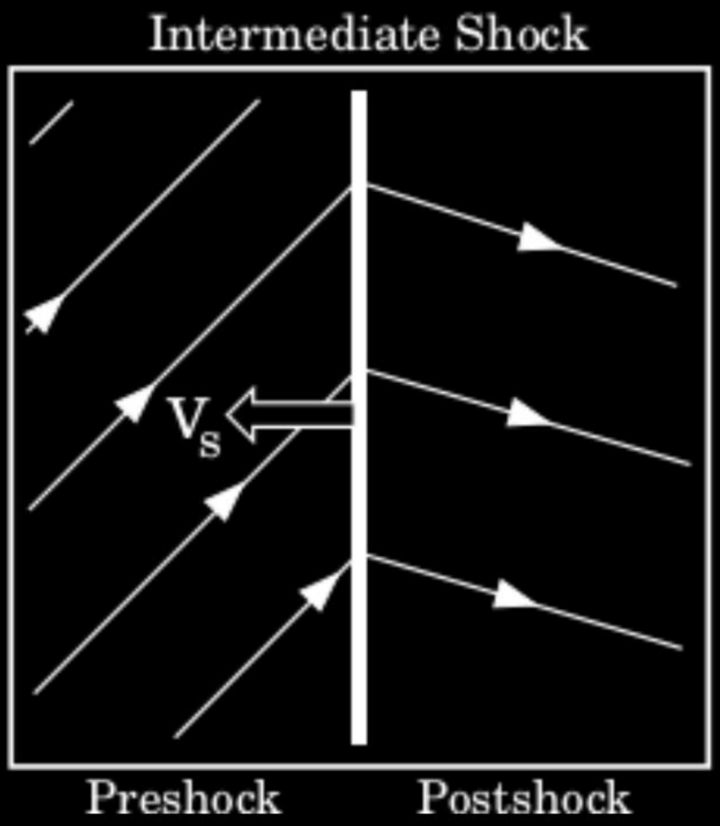
6.2 SHOCKS

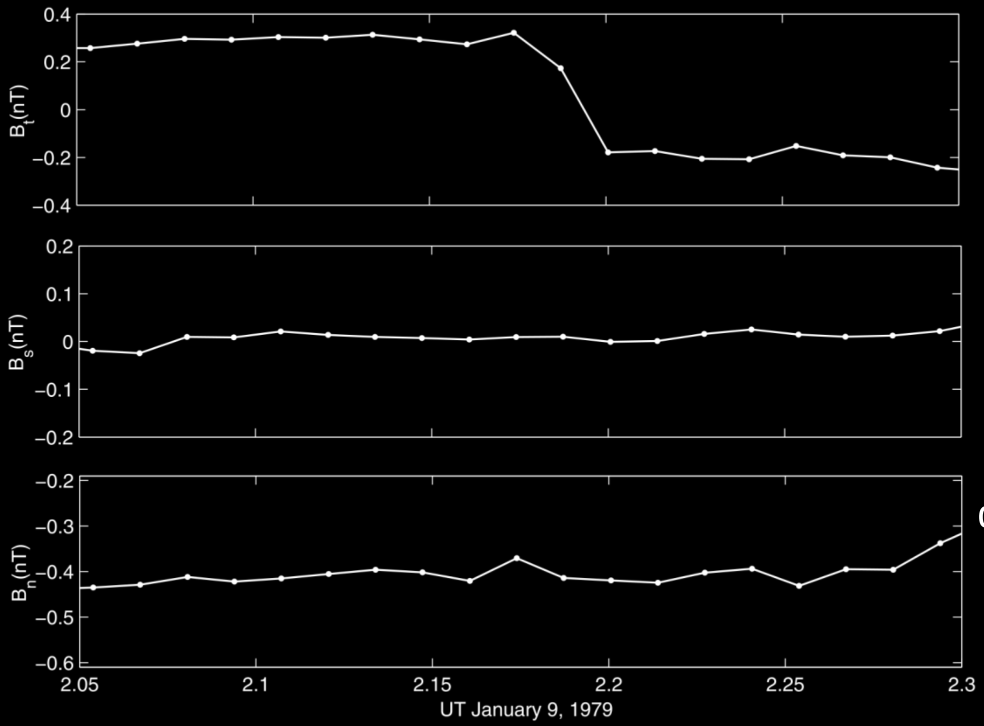
2c. Intermediate shocks

$v > v_A$

$v < v_{fast}$

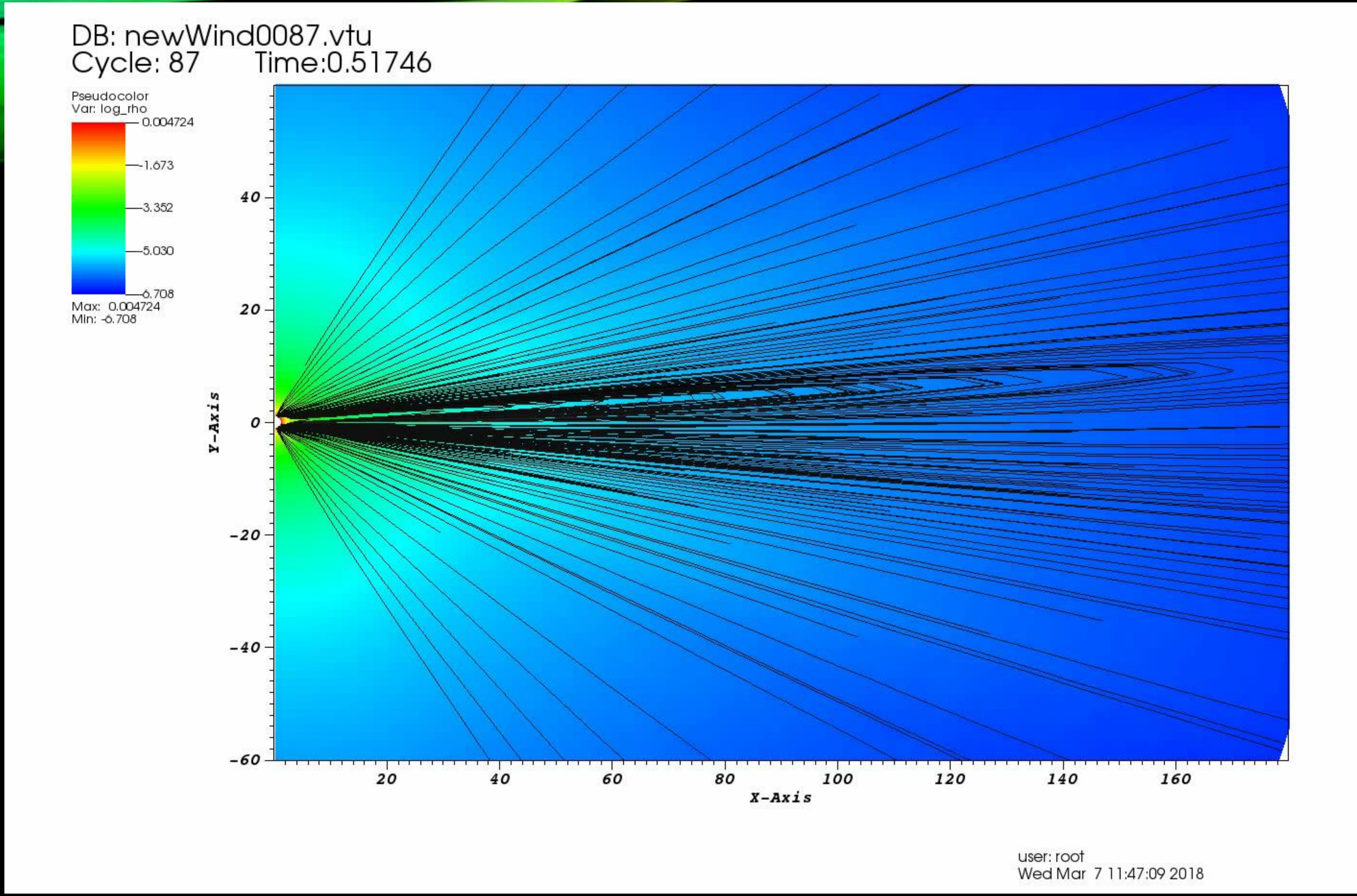
→ The tangential component of the magnetic field flips across the shock normal





Observed magnetic field on 9 January 1979 in the shock coordinate system (intermediate shock). Credits: Feng and Wang, 2008

Feng and Wang (2008) identified an intermediate shock observed by Voyager 2 on January 1979. The tangential component of the magnetic field changed sign across the shock front; the normal Alfvén Mach number is greater than unity in the preshock state and less than unity in the postshock state; the fast-mode Mach numbers in the upstream and downstream regions are less than unity and both slow-mode Mach numbers are greater than unity → intermediate shock.



MPI-AMRVAC simulation of 2 CMEs until 1 AU

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and some nice presentations: https://www.cfa.harvard.edu/~namurphy/teaching.html

ARIGATO FOR YOUR ATTENTION!

AND DON'T FORGET….

…MHD IS AWESOME!